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Pre-service Mathematics Teachers' Mental Constructions of the Determinant Concept

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ABSTRACT This study aims to reveal the nature of mental constructions made by pre-service teachers when learning the determinant concepts and to contribute to APOS theory (Action, Process, Object and Schema) in terms of instructional strategies. This is guided by the belief that understanding the mental constructions the students make when learning mathematical concepts leads to improved instructional methods. The data was collected through a structured activity sheet and interviews. The findings revealed that the mental constructions made by pre-service teachers in most cases do concur with the preliminary genetic decompositions, and that many pre-service teachers are operating at an action/process stages, with the exception of the few operating at an object stage. It also revealed that many students could carry procedures effectively, even if the meaning of the concept is not constructed, which indicates that students mainly possess procedural knowledge of the determinant concept.

INTRODUCTION

The focus of teaching mathematics to undergraduates students aims for them to achieve an understanding of the interrelationship between concepts, rather than simply carrying out procedures. However, research has shown that in many cases, students who reach universities experience difficulties in making such connections, because, for most of their schooling career, the focus in teaching of Mathematics has been on learning that encourages students to build knowledge specific to a particular problem (Hong et al. 2000). Students construct mathematical knowledge as isolated facts, where as a result, they assimilate rules cognitively as a list of unconnected actions. For example, in school algebra, students tend to associate the word function only with graphs. They do not think of multi-representational functions, such as tables and equations. This paper aims to explore the nature of mental constructions that students are able to make when learning determinants as a concept. Determinants are an essential part of matrix algebra. They are applicable in many contexts, such as in solving the system of equations, matrix inverse and analytic geometry. Therefore, knowledge of determinants is used and required throughout the whole course of matrix algebra. The authors have already studied pre-service teachers' mental constructions when learning calculus, fractions and matrix algebra, respectively (Brijlall 2014a, c; Ndlovu 2014; Brijlall and Ndlovu 2015; Brijlall and Maharaj 2015). The understanding of the nature of the mental constructions that students make and their association in the development of conceptual understanding of mathematical concepts is of great importance in developing and implementing appropriate instructional strategies. Dubinsky (2010) has claimed that constructing the appropriate mental structures for a given mathematical concept leads to the easy learning of the concept. Therefore, pedagogy should aim at helping students build relevant mental structures.

Findings from this research could lead to more effective teaching practices, especially in matrix algebra classes, as mathematics lecturers will know the level of knowledge the students have of the concept and therefore design the teaching in a manner that will help students to construct the necessary knowledge. It will also contribute to the development of an in-depth student understanding of determinant concept, contributing new knowledge to the mathematics community, as it will introduce new genetic decomposition for the determinant concept. Moreover, the processes students undergo to build their knowledge when learning these concepts will be identified and analysed, for better understanding of the knowledge constructed. The description of how these mental constructions are connected for a particular mathematical concept is called genetic decomposition. For the purpose of this paper, the preliminary genetic decomposition of the determinant concept is illustrated in Figure 2. The nature of mental constructions made is explored through the following research questions:

What is the nature of pre-service teachers mental constructions of the determinant concept?

To what extent, if any, do the students' mental constructions of action-process-object link with preliminary genetic decomposition?

What characteristics of the schema displayed by pre-service teachers can be adopted to modify the preliminary genetic decomposition?

Literature Review

Linear algebra is one of the first courses of study undergraduate students come across in advanced mathematics. Wawro et al. have noted that "it is one of the most useful fields of mathematical study because of its unifying power within discipline and applicability to areas outside pure mathematics" (2011: 2). Therefore, it is imperative that students are able to recognise and make the necessary connections between the taught concepts so that they could apply it to other context and be able to solve related problems. The goal of mathematics instruction is to help students develop a well-organised collection of versatile mathematical procedures that they can call upon to solve problems in a variety of situations (Hasenbank 2006). Unfortunately, many students that reached the university have not yet developed these skills, but instead have mastered a collection of rules and algorithms, which in most cases, become a barrier in their learning, as they encounter difficulty in sorting through them for the appropriate approach to use for a particular problem. Durkaya et al. (2011) investigated first year pre-service teachers' pedagogical content knowledge of a determinant concept. Their study revealed that many students have trouble in taking determinants, and could not express the definition and the meaning of the determinant, indicating the lack of learning for meaning. This concurs with what Naidoo (2007) has alluded to, that students rely on rules and algorithms, and as a result, they struggle to solve unfamiliar problems. This is probably because in most cases, algorithms are mainly memorised, lacking what we called mathematical proficiency. In other recent studies, Brijlall (2014b) and Brijlall and Maharaj (2015) studied the pedagogical content knowledge necessary for high school mathematics pre-service and in-service teachers. Ndlovu and Brijlall (2015) explore the students' mental constructions of matrix algebra. The findings in those studies propagated further explorations into pre-service teachers learning mathematics. This paper addresses such a need.

In order for proper mathematical learning to take place, procedural knowledge needs to be connected to conceptual learning. Hobden (2006) defines conceptual understanding as an integrated and functional grasp of mathematics, resulting in the ability to see connections between ideas, and a 'big picture' of procedures. Thus, when students have conceptualised a concept, it would mean that they can identify the link between concepts and adopt different heuristics that can be useful in a particular problem for a particular context, and be able to link the facts in mathematics. For example, besides just using the formula to solve the determinant of a matrix, it is imperative that students understand the meaning of the determinant and its application to other concepts.

Theoretical Framework

This study is underpinned by the framework for research and curriculum development in mathematics education, as advocated by Asiala et al. (2004), which focuses on cognitive growth in the way in which students construct mathematical knowledge. The framework consists of three components: theoretical analysis, design and implement instruction, observations, and assessments (Fig. 1). Asiala et al. have noted that "the theoretical analysis produces assertions about mental constructions that can be made in order to learn a particular mathematical topic, instruction tries to create situations which can foster making these constructions, observation and assessment tries to determine if the constructions appear have been made and the extent to which the student actually learned." Figure 1 shows the research framework for research and curriculum development. It is with this research framework that the theoretical framework adopted from Brijlall et al. (2013) fits theoretical analysis.

In this section, the researchers will consider theoretical analysis, with a focus on mental construction. Under theoretical analysis, the APOS

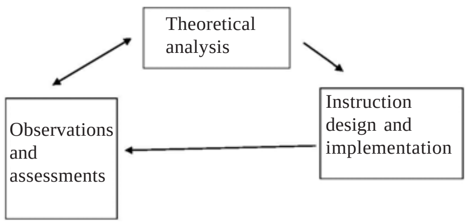

Fig. 1. The research frame work

theory of Dubinsky will be useful, by which he describes the cognitive structures used by students to construct knowledge through action, process, object and schema.

Theoretical Analysis

Under theoretical analysis, this study employed APOS theory (Action-Process-Object-Schema) to describe and analyse pre-service teachers' knowledge of matrix algebra concepts constructed. The theory will be discussed in detail later. The aim of applying APOS theory was to reveal the nature of students' mental constructions, not to provide statistical comparison of pre-service teachers' performances in mathematical concepts. APOS theory proposes that an individual has to have the appropriate mental structures relating to action, process, object and schema in order to make sense of the given mathematical concept. These mental structures need to be detected, and then learning activities that will be suitable for the development of those mental structures ought to be designed to enhance the construction of those mental structures (Maharaj 2014). This led to the design of a genetic decomposition relative to specific mental constructions that students make, in order to develop their understanding of matrix algebra concepts. Under theoretical analysis, the preliminary genetic decomposition was designed for the determinant concept, relative to specific mental constructions that an individual might make. Then the instructional design followed in the form of class activities focusing on addressing the mental constructions indicated in the preliminary genetic decomposition. The last step was to implement the instruction through data collection. The theoretical perspective adopted for the study was also used to analyse the data collection, and observation was made regarding which mental constructions were constructed by students. The complete description of the research framework used in the study can be found in Asiala et al. (2004).

Preliminary Genetic Decomposition

Theoretical analysis considers how students cognitively construct knowledge in mathematics. Therefore, it is necessary to design instruction that would allow students to make such constructions. Such design is called a genetic decomposition, and a number of South African studies have designed the genetic decomposition for mathematical concepts in calculus (Brijlall et al. 2011; Ndlovu 2012; Brijlall and Ndlovu 2013; Jojo 2014; Maharaj 2014; Brijlall and Maharaj 2015). Few studies however, have explored the design of genetic decomposition for matrix algebra concepts (Ndlovu 2014; Ndlovu and Brijlall 2015).

A genetic decomposition refers to the structured set of mental constructs, which might describe how the concept can develop in the mind of an individual (Brijlall et al. 2013). As Dubinsky (1997) points out, constructing a genetic decomposition of a concept does not mean that learning of a mathematical concept follows a single route, or that is the only way it can be learnt. However, it helps with observing of the learning in progress, and it is a guide for one possible way of designing instructions. The preliminary genetic decomposition designed was based on the researchers' experiences of the particular concept. It did not necessarily represent how trained mathematicians understand concepts. What shall follow in Figure 3 is the proposed genetic decomposition of the concept of matrix algebra.

Instructional Design and Implementation

The preliminary genetic decomposition of the determinant concept given here guided the researchers' teaching instruction in class and the design of activity sheets. The activities were designed to assist students in making the suitable mental constructions as indicated in the preliminary genetic decomposition. In the theoretical analysis, the genetic decomposition indicating how student can construct the knowledge for determinant concepts was presented. Following such, the design activities aimed at

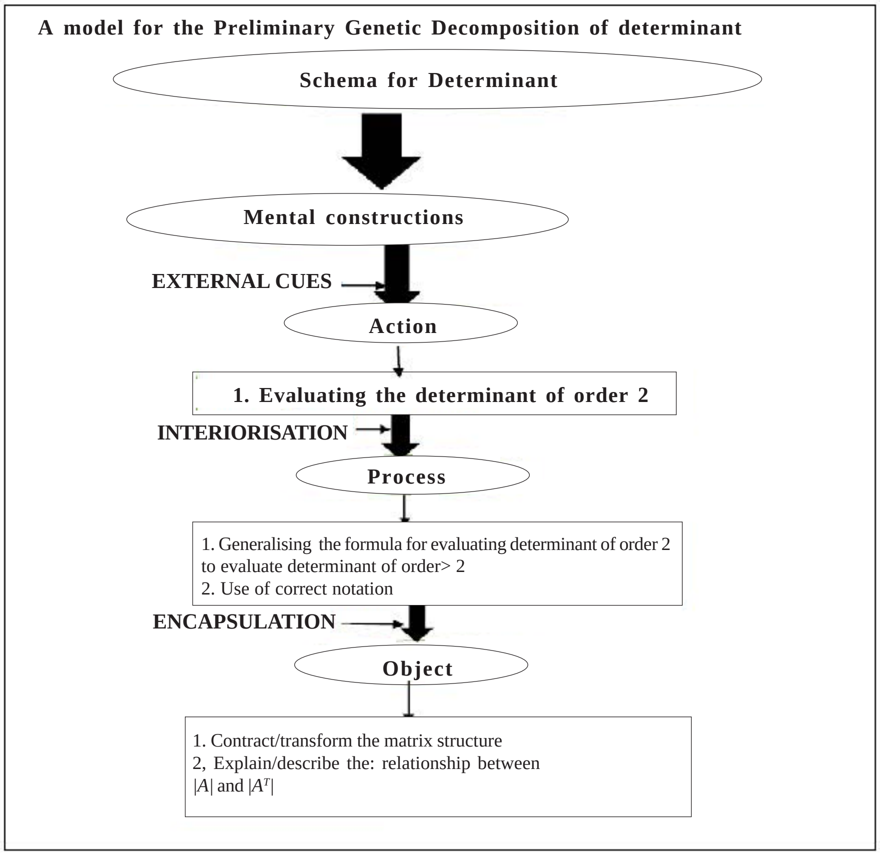

Fig. 2. A preliminary genetic decomposition for determinants

getting students to make the constructions were prepared. This followed the Activity-Class-Exercises (ACE) teaching style. The activities were prepared in such a way that students would complete them in 90 minutes. After each task was completed, a class discussion took place, allowing students to reflect on the work they had undertaken.

Observation and Assessment

This allowed the researchers to gather and analyse data. The results were used to reveal the nature of students' mental construction and how these concur with the preliminary genetic decomposition. The students' responses from the structured activity sheets were used in designing relevant interview questions. This was done with the aim of observing the mental constructions made, so as to assess the processes they undergo in building the knowledge, which will reveal the level at which they are at in terms of APOS.

METHODOLOGY

To answer the research questions, a qualitative approach was adopted. Qualitative research methods involve the systematic collection, organisation, and interpretation of textual or verbal data (Bertram 2004). These methods are used in the exploration of meanings of social phenomena, as experienced by individuals themselves, in their natural context (Malterud 2001). For the purpose of this study, data was collected through structured activity sheet. The tasks chosen were those that the researcher identified as suitable to allow the pre-service teachers to make the necessary mental constructions, as explained in the preliminary genetic decomposition. When solving the tasks, pre-service teachers were asked to work individually, and were given two days to complete the tasks. The tasks were collected and marked by Researcher One. In this study, data was generated from students' interpretations of mathematical problems, and their solution to the given problems. It is through analysing the generated data that the proposed genetic decomposition hoped to yield insight and understanding into how the subjects make meaning of the learnt concepts. The mental constructions made were analysed through the set of meanings proposed in the genetic decomposition, thereby allowing the development of the theory. The analysis of the written responses led to the improvement of the tasks in the activity sheet and a modified genetic decomposition. It is hoped that this modified genetic decomposition will inform our pedagogy when addressing the pedagogical knowledge of these pre-service teachers (Brijlall 2014; Brijlall and Maharaj 2015).

Since the study was based on a qualitative approach, both inductive and deductive analyses were used. This was done through coding, as well as the written responses of eighty-five pre-service teachers. Once coded categories were identified, the responses were grouped accordingly. These categories assisted in identifying different components of APOS in various types of correct and incorrect solution strategies. Thereafter, patterns that emerged were identified and discussed. Moreover, the preliminary genetic decomposition presented in Figure 1 served as an analytic tool, it helped in describing the mental constructions made by pre-service teachers. After analysing the worksheet, meaning was assigned to their responses and the level at which they were operating was described in terms of APOS theory.

RESULTS AND DISCUSSION

The analysis in this study was based on students' responses to activity sheets and transcripts from the interviews. Out of 31 participants, five were selected for the interviews. For this study, it was of importance to explore whether they could evaluate and apply determinants to solve other related problems. The activity sheet comprised three items. Item 1 aimed to explore pre-service teachers' knowledge of evaluating determinant of a matrix. It addressed the action and process levels of the preliminary genetic decomposition in Figure 2, and was intended to provide insight into whether or not the students had developed the action and the process conception of evaluating determinants, and a correct use of notations. To be able to evaluate determinants of 3×3, students needed to have understood the procedure of solving 2 2, understand the meaning of the matrix order.

Item 1

Given C= $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 3 & 1 \\ 5 & -2 & 8 \end{bmatrix}$ D= $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -4 & 0 \\ -2 & 5 & -3 \end{bmatrix}$

1.1 Find the determinant of C 1.2 Find the determinant of D^T

In Table 1 the allocation of scores for item 1 is displayed.

Table 1: Allocation of scores for item 1

Score	*1*	*2*	*3*	*4*	*5*
Indicator	No response or incorrect response in both questions.	Shows some knowledge of the formula but failed to carry out procedures correctly.	One response is correct and one is totally incorrect or partially correct.	Correctly evaluated the determinants in both questions but used incorrect notation to express the answer.	Complete and ccurate a response.
Number of students in each category	6	5	18	0	2

Only six percent of the students gave a complete response to Item 1, indicating complete mathematical understanding of evaluating determinants. Their responses indicated that they had constructed the necessary mental constructions, as expected in the preliminary genetic decomposition. This the researchers observed as they showed that they had the collection of all the procedures, and could apply them in all related context. Their responses showed that they had constructed the correct concept image of the concept. This was revealed as they showed clear understanding of the relationship between the minors and cofactors, and used this to evaluate the determinants of the matrices. Their responses showed interiorisation of actions into the process that generalised the formula of order two to higher order, and showed possibilities for coordinating the relationship of the matrix transpose and determinant of a matrix. Based on that, it seemed that the process conception of the concepts was fully developed. The responses of six students in category one showed that they had more difficulties in evaluating determinants, indicating that the action level had not yet developed. This the researchers observed as they failed to manipulate numbers and to generalise the formula for evaluating determinants of order two, to evaluating higher order determinants. This meant that the mental constructions necessary to understand the concept had not developed.

Fifty-eight percent of the students provided the correct response in only one of the two items. Sixteen out of the eighteen students provided a correct response to Item 1a, and an entirely incorrect response to Item 1b. The other two provided a correct response to Item 1b, and a partially correct response to Item 1a. The two responses to Item 1b were considered partially correct, because from their responses, it seemed that they knew the procedures to follow, but failed to manipulate numbers, and therefore, provided incorrect responses. There were a number of errors identified from the responses of the sixteen students who provided a correct response to Item 1a and an entirely incorrect response to Item 1b. Some used the minors as cofactors or substituted incorrect cofactors. Some failed to manipulate the signs, ignoring the effect of multiplying by a negative number, and some did not even attempt to solve Item 1b. Their responses or lack of response indicated that the schema of basic algebra had not developed, and had impacted negatively on the new knowledge learnt. This confirms what Dubinsky (1997) pointed out, namely that students' difficulties with linear algebra emanate from the lack of background knowledge of school algebra and arithmetic algebra. The same was echoed by Ndlovu and Brijlall (2015), namely that the lack of related schema such as real number system impacted negatively on the construction of matrix algebra schema for some students. Zama's response to Item 1b indicated a number of shortcomings other than the ones mentioned above (see Fig. 3).

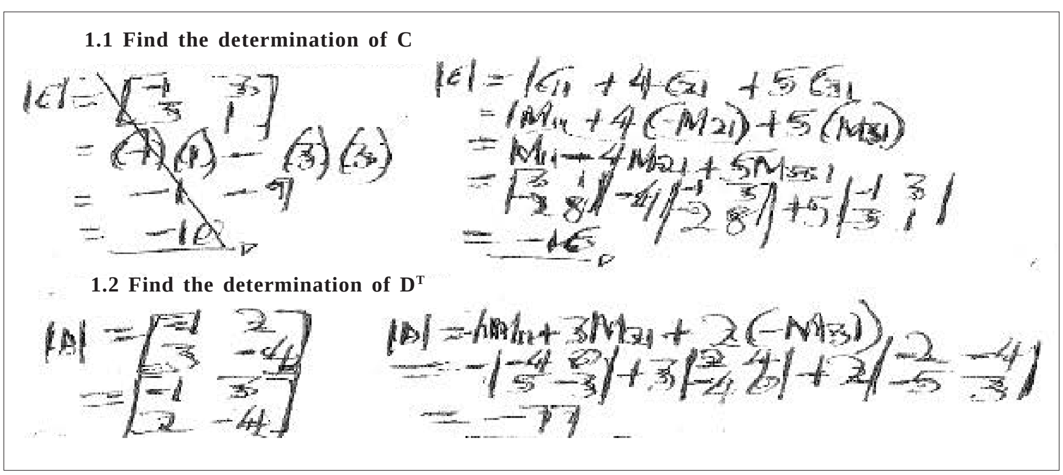

Fig. 3. Zama's response to Item 1

She was unable to determine the transpose of D. Examining her response, she first deleted D_{33} and had a sub-matrix of order two in the form. Then she interchanged some entries of the sub-matrix to be in the form This shows that she had not fully interiorised the action of evaluating determinants of order two. This the researchers observe more as she confused the formula of determining the determinant and that of determining the matrix inverse of order two, which she also did not apply correctly. This shows that when ideas are not cognitively constructed, they end up becoming muddled, and cause more confusion on the side of the students. Moving on the right hand, we see her attempting to evaluate the determinant of D by following the same procedure as in Item 1a. In finding she expanded by Column 1 and incorrectly determined the minors and cofactors (see the middle and last minors in Fig. 3). She also failed to manipulate numbers correctly to determine the determinant. Her response indicated that she is operating at the action stage in terms of APOS. This, we observed as she correctly determined the determinant of Matrix C, and with some difficulty, tried to generalise the formula to evaluate but failed to manipulate the entries. The researchers also observe that she evaluated instead of which shows that she coordinated the relationship between the determinant of a matrix and the determinant of its transpose. Here are some of her responses:

Researcher: In which type of matrices can you evaluate the determinant?

Zama: In square matrices.

Researcher: How do we evaluate the determinant of order?

Zama: [silent].

Researcher: In Matrix C, which row or column you expanded with?

Zama: Column 1.

Researcher: If you expanded by any other column or row would the answer be different?

Zama: I do not think so.

Researcher: Can you explain how you determine?

Zama: I first find the minors by deleting some rows and columns, and then work out the determinant, because it was easy now, since these matrices were 2 x 2 and there is a formula for that.

Researcher: Would the same procedure work for?

Zama: Not sure, since I never done [sic] something like this before.

Zama's responses shows that she does not have an understanding of the meaning of her solution. The action of computing determinants has not been interiorised into a process. This we observed as she failed to recognise that same procedure of evaluating determinants is also applicable in evaluating determinant of the matrix transpose; and meant that she was still operating at the action level.

It was further noted that a number of students used incorrect notation when evaluating determinants. For example, while evaluating students wrote instead (see Fig. 4). Like Zama, Thabo also provided the correct response for Item 1a, but did not even attempt to evaluate the and used incorrect notation when determining the determinant of matrix C. Instead of he wrote and the same also when indicating the element of the Matrix C in the formula he used $a_{31,}$ suggesting weak understanding of the meaning of the notation used in determinants. This show that students could carry out procedures without constructing the meaning of the concepts and, as Maharaj (2014) has pointed out, students do not interrogate what they write. Here we present Thabo's response to activity sheet followed by an interview extract.

Researcher: You correctly evaluated the determinant of Matrix C, why did you not evaluate ?

Thabo: I was slow by the time we had to do group discussion, I was still solving the determinant of Matrix C, but I was not worried, because I knew what to do.

Researcher: In evaluating the determinant of Matrix C you expanded with Row 3; if you expanded by any row or column, would your answer have changed?

Thabo: It will be the same; just different numbers in the equation, because all what we need is to find the sub-matrix of the original matrix, so it does not matter whether I find it in one particular row or column, or another, as long as I am consistent about the row or column that I used.

His response clearly indicated that Thabo had done this at an action level repeatedly, and now had interiorised the action to a process. He could mentally think about how the concept links. This the researchers observed as he clearly explained why using any row or column does not change the answer, indicating the evolution in the constructions related to the determinant and its application. This shows he is not just manipulating numbers or rules, but that instead,

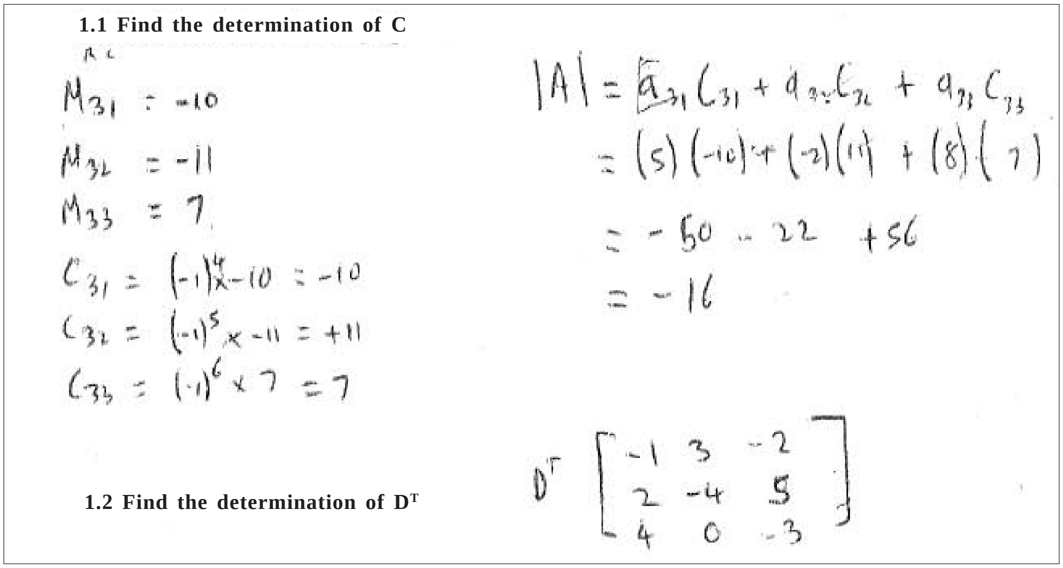

Fig. 4. Thabo's response to item 1

he is actually thinking about a process as a whole. The development of basic algebra schema assisted to accurately compute the determinant.

Researcher: When evaluating the determinant of matrix C, you wrote . Why?

Thabo: I am so used working with examples as the determinant of A, it was supposed to be.

Researcher: When evaluating the determinant of Matrix C in your formula, you wrote a_{11}. What is the meaning of this?

Thabo: Was this also need to change [sic]. I did not know how to write $c_{11,}$ since there was also c_{11} for a cofactor.

His interview revealed that he did not recognise the notation, as the standard mathematical notation that constructs meaning of the concept. Instead, he saw them as letters that he could merely interchange so as to lessen the confusion, meaning that the understanding of the notation is not cognitively constructed. It seemed that the notational distinctions that he was making were not necessary the ones considered to be standard, and focusing on the meaning of the concepts. These results coincide with the findings in the literature that learning new notation may be better seen by some students as accommodation rather than assimilation (Findell 2006). Findell (2006) also pointed out that learning a term or notation requires building some cognitive structures around the notation to support its meaning and use. Rowland et al. (2016) further note that it is a necessity that students constructs rationality and semantic of the concepts learnt in order to communicate it effectively. This indeed has huge implications for the teaching and learning of mathematical notation. The meaning of notation and the context it is used ought to be emphasised for the conceptual understanding of the meaning of the determinant and other mathematical concepts.

Item 2

Item 2 was designed to explore students' understanding of the determinant of a matrix and determinant of its transpose. It aimed to provide insight about students' conceptual knowledge in relation to matrix and transpose, and to address the process conception of determinants, as indicated in the preliminary genetic decomposition.

Without doing any calculation, what is the determinant of D? Explain your reasoning.

In Table 2, the allocation of scores for Item 2 is displayed.

Twenty students provided a correct response, indicating that they had made the connection between the determinant of the matrix and determinant of its transpose. Seven of these students showed the evolution of a pre-deter-

Table 2: Allocation of scores for Item 2

Score	*1*	*2*	*3*
Indicator	No response or incorrect response	incorrect response with no explanation	Correct response with clear explanation
Number of students in each category	10	1	20

minant schema as they interpreted the question and presented their solution clearly. Their explanation showed that these students could switch from actions, to doing mathematics, to concepts by means of which to think about meaning. This revealed that they had made necessary mental construction of the concepts as indicated in the preliminary genetic decomposition. When asked to explain the relationship between determinant of a matrix and determinant of its transpose, Thabo said: it is the same, because the determinant of a matrix equals the determinant of its transpose. If we find the determinant, we expand by a row or a column, and if we find a transpose, we exchange the rows with the column. That means a column in a matrix is a row in the transpose. As you see here [pointing at his solution of the determinant of Matrix C] I expanded by Row 3, which will be Column 3 in the transpose; so it will give the same answer there if I expand by it, because when we find determinants, we can expand by any row or column.

Thabo must have done this at an action level repeatedly, until he internalised it, as he realised that answer remains the same regardless of the row or column used. His response showed that he was able to coordinate the process of evaluating determinants and determining transpose to form new processes. He cognitively constructed the collection of the concept and could recognise the relationship between the necessary concepts.

Thirteen of these students had some difficulties in articulating the relationship clearly using correct mathematical language. For the development of these concepts, students need to construct the appropriate use of language and terminology, as well as notations. The group of students who had the weakest understanding of the relationship between the determinant of a matrix and determinant of its transpose, demonstrated a lack of ability to interiorise the action into process. Zinhle's wrote that "where there is + there will be – in D". It seemed the word transpose to her referred to changing the sign, because her determinant of the transpose was one hundred, triggering a misconception she may have been harbouring, which caused a barrier in making the necessary mental constructions. It is possible that what has been learnt in school algebra has been incorrectly generalised, because it was never understood, but memorised. In school algebra, when solving equations, students are told to transpose the number over the equal sign, and to change the sign. This might have created the misconception, where the term transpose becomes associated with changing the sign. In matrices, she learnt that the transpose means interchanging the columns, therefore she cognitively constructed the meaning with which the word transpose is associated, changing something, and in this case, linking it with changing the sign as she was taught to do at school. According to Aygor and Ozdag (2012), students have a tendency to over-generalise a correct conception. Zinhle's prior and current understanding of the word transpose impacted negatively on the construction of appropriate knowledge of the concept. This is what Tall (2008) has respectively referred to as 'met before' and 'met after', which he says if concepts learnt earlier or after a particular concepts ar not properly understood, may cause barriers to in constructing conceptual understanding of the current taught concept. In this case the conceptualization of the determinant.. During the interview with Zinhle, the following conversation took place.

Researcher: What do you mean by saying where there is plus there will be minus?

Zinhle: Actually, I did not have a full understanding of how the determinant of D would differ from the transpose. What I meant was where the cofactor is positive, D will be negative.

Researcher: How do you determine the transpose of a matrix?

Zinhle: The transpose is a... is the opposite of... just exchange the rows with columns.

Researcher: When evaluating determinant of Matrix C, you expanded by Column 1. If you were asked to find the transpose of Matrix C, what was Column 1 going to be in the transpose?

Zinhle: Column 1 will be Row 1.

Researcher: If then you are were asked to find the determinant of the transpose of Matrix C, and you expanded by Row 1, what was going to be your determinant of the transpose?

Zinhle: I do not know.

Researcher: But you are expanding by Row 1 in the transpose, which was Column 1 in the matrix, so that means you are using the same entries. So what does that tell you about the determinant of the two?

Zinhle: *Oh man!* [sic] It will be the same.

After being probed several times she seems to understand that the two determinants are the same, however, her responses do not reveal whether or not she understood clearly what a transpose is. Her conception of the word transpose as meaning opposite seems to cause her difficulty, where as a result, she could not construct the necessary mental construction to make sense of the concept.

CONCLUSION

The researchers' proposition is that for the evolution of conceptual development of the determinant concept, students need to have at least developed the process conception of the concept. This paper has provided answers to the researchers' three research questions. The nature of mental constructions made by pre-service teachers evaluating determinants of order was considered to be at the process stage in terms of the presented preliminary genetic decomposition. To carry out the required necessary procedure and construct necessary mental constructions, an individual needed to know the formula of evaluating 2 x 2 determinants, which is considered to be at the action stage. Evidence from Item 1 (evaluating determinants) revealed that only 6 percent of students had interiorised the action of evaluating determinant into a process which indicated that only two pre-service teachers understood the procedures of evaluating determinant of order to a point that they could explain the connection made between their general statement of evaluating determinant and its applicability to other contexts. The researchers therefore conclude that as they progressed with the course, their knowledge of determinants became more sophisticated, and showed some conceptual evolution as they recognise the relationship between procedures of

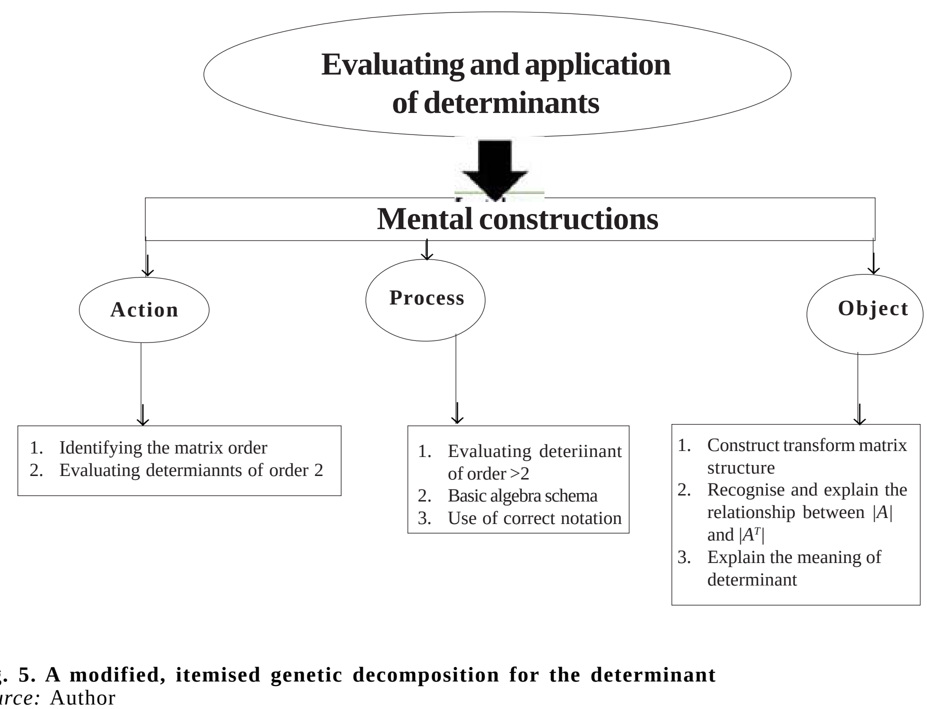

Fig. 5. A modified, itemised genetic decomposition for the determinant
Source: Author

evaluating determinants of order two to determinants of order

RECOMMENDATIONS

The findings of this study suggest that the action conception of the determinant has fully developed, while the process stage was still developing for the majority of students. This meant that although many students could carry out procedures, they had not constructed the meaning of the concepts. Some students still lacked the evolution in the constructions related to determinants concepts, as they experienced difficulties in unpacking the structure of. This meant that they had not made the connection between the determinant of the matrix and determinant of its transpose. Only few students seemed to have encapsulated the process stage into an object, as they could apply procedures to problems in unfamiliar context and make meaning of their solution.

Secondly, the results revealed that preliminary genetic decomposition proved to be useful in assisting us to identify the level at which the pre-service teachers were operating in terms of APOS, which meant the mental constructions made do link with the mental constructions as indicated in Figure 2. The preliminary genetic decomposition also proved useful in assisting the researchers in identifying the ideas of a determinant that had not fully developed. For example, it was noted that although many students could carry out procedures, the meaning of the concepts was not conceptually understood. Most students could not evaluate, and some failed to generalise the formula for evaluating determinant of order two to evaluating determinant of order. Thirdly, although preliminary genetic decomposition proved to be useful, some aspects considered to be necessary for conceptual evolution of the determinants were not included, such as basic algebra schema and constructing meaningful understanding of the concept. These aspects are deemed necessary for the conceptual understanding of the determinant and its application to other concepts. It is for this reason that the researchers recommended the modification of the model.

The researchers present the modified genetic decompositions in an itemised manner in Figure 5 accordingly.

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