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ORIGINAL RESEARCH

Algebraic and Singularity Properties of a Class of Generalisations of the Kummer–Schwarz Equation

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Abstract The Kummer–Schwarz Equation, $2y'y''' - 3y''^2 = 0$, (the prime denotes differentiation with respect to the independent variable x) is well known from its connection to the Schwartzian Derivative and in its own right for its interesting properties in terms of symmetry and singularity. We examine a class of equations which are a natural generalisation of the Kummer–Schwarz Equation and find that the algebraic and singularity properties of this class of equations display an attractive set of patterns. We demonstrate that all members of this class are readily integrable.

Keywords Kummer–Schwarz · Symmetries · Singularities · Integrability

Mathematics Subject Classification 34A05 · 34A34 · 34C14 · 22E60

Introduction

The Kummer–Schwarz Equation [11],

$$2y'y''' - 3y''^2 = 0, \tag{1}$$

is notable amongst the class of third-order ordinary differential equations due to its properties as a differential equation apart from its well-known connection to the Schwarzian Derivative.

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Some of its properties were discussed in [9] and we briefly recall them here. Equation (1) possesses six Lie point symmetries, namely

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x, \quad \Gamma_3 = x^2\partial_x, \quad \Gamma_4 = \partial_y, \quad \Gamma_5 = y\partial_y, \quad \Gamma_6 = y^2\partial_y. \tag{2}$$

The algebra is a double dose of $sl(2, R)$ with the first three symmetries providing one representation and the last three symmetries providing the second representation. The full algebra is $sl(2, R)\oplus sl(2, R)$. In terms of contact symmetries it has ten with the algebra $sp(5)$ [1] and so may be transformed to the archetypal third-order equation of maximal symmetry, $y''' = 0$, by means of a contact transformation. The equation also has attractive singularity properties with a simple pole and resonances at $-1,\ 0,\ 1$. The solution,

$$y = \frac{A_1}{x + 2A_0} + A_2,$$

reflects the singularity properties nicely.

The Kummer–Schwarz Equation is a representative of a general class of equations which can be written as

$$y^{(n-2)}y^{(n)} - my^{(n-1)^2} = 0, \tag{3}$$

where m is a parameter and n is an integer of lowest value 2, *ie*, the class of equations actually starts at the second order.We provide the algebraic and singularity properties and solutions for a number of the earlier elements of the class. From these results we infer general properties, state them as conjectures and then prove these conjectures.

Symmetry Properties of the Class of Kummer-Schwartz Equations

We examine Eq. (3) for its symmetry properties. If $n = 2$, there are eight Lie point symmetries given by[1]

$$\begin{aligned}
\Gamma_1 &= \partial_x\\
\Gamma_2 &= x\partial_x\\
\Gamma_3 &= y\partial_y\\
\Gamma_4 &= xy\partial_y\\
\Gamma_5 &= \log(y)\partial_x\\
\Gamma_6 &= y\log(y)\partial_y\\
\Gamma_7 &= x^2\partial_x + xy\log(y)\partial_y\\
\Gamma_8 &= x\log(y)\partial_x + (y\log(y))^2\partial_y
\end{aligned} \tag{4}$$

for $m = 1$. Equally for $m \neq 1$ there are eight Lie point symmetries. Now the value of m intrudes into the expressions for some of the symmetries. The symmetries are

$$\begin{aligned}
\Gamma_1 &= \partial_x\\
\Gamma_2 &= x\partial_x\\
\Gamma_3 &= y\partial_y\\
\Gamma_4 &= y^m\partial_y\\
\Gamma_5 &= xy^m\partial_y
\end{aligned}$$

[1] For the calculation of the symmetries we use the Mathematica add-on Sym [3–6].

$$\Gamma_6 = \frac{y^{1-m}}{m-1}\partial_x$$
$$\Gamma_7 = (1-m)x^2\partial_x + xy\partial_y$$
$$\Gamma_8 = (1-m)xy^{1-m}\partial_x + y^{2-m}\partial_y. \tag{5}$$

Because the maximal number of Lie point symmetries for a scalar second-order ordinary differential is eight [10, p. 405] for $n = 2$ the algebra is $sl(3,\ R)$ irrespective of the value of m.

We give the symmetries for $n = 2$, above, separately to those of $n > 2$, below, due to the peculiar types of symmetry to be found in the case of the former for, in addition to the usual solution symmetries, homogeneity symmetry and elements of $sl(2,\ R)$, there are the two noncartan or fibre-preserving symmetries [8] given by Γ_7 and Γ_8.

When one moves to higher values of n, the situation changes and we give the results for a few low values of n to enable one to have a feel for the results to be proven below. We highlight the various symmetry results in the table below for the values of n, 3, 4, 5, 6 and 7. The listing of the symmetries is ordered as solution symmetries, homogeneity symmetry and such elements of $sl(2,\ R)$ as persist when the number of point symmetries is less than maximal. The Kummer–Schwarz Equation $(n,\ m) = (3,\ \frac{3}{2})$ is also somewhat exceptional in that there is a doubling of the $sl(2,\ R)$ subalgebra. The subalgebra of $sl(2,\ R)$ corresponding to the one found for $y''' = 0$ is that based on the ∂_x symmetries. This becomes obvious when one looks at the results for $(4,\ \frac{4}{3})$, $(5,\ \frac{5}{4})$ and $(4,\ \frac{6}{5})$. The coefficient of $xy\partial_y$ decreases with n and is naturally zero for $n = 3$. What is really exceptional is the persistence of $y^2\partial_y$ one finds in one of the noncartan symmetries of $y'' = 0$.

In the illustrative examples given in Table 1 a pattern emerges for the possible sets of symmetries of equations of the form of (3). For $m = 0$ the number of symmetries is the same as for $y^{(n)} = 0,\ n = 3,\ 4,\ 5,\ 6, 7$. The general result for this class has been given in [12] and the algebra is $\{A_{3,8} \oplus A_1\} \oplus_s nA_1$ in which the nomenclature of the Mubarakzyanov Classification Scheme [13–16] has been used for the subalgebras. For general values of the parameter, m, there appear to be $n+1$ symmetries from the examples considered. For an nth-order linear equation there are three possibilities reported in [12]. The maximal case has $n+4$ point symmetries. The other cases are $n+1$ and $n+2$. The former comprises n solution symmetries and the homogeneity symmetry, $y\partial_y$. The latter has these plus ∂_x indicating autonomy. However, here we do not have the same collection of symmetries. There are solution symmetries of an equation of order two less than the one under consideration, the homogeneity symmetry and a subset of two of the three elements of $sl(2,\ R)$, (*ie* $A_{3,8}$) to be found in the equation of maximal symmetry. The algebra is $2A_1 \oplus_s \{A_1 \oplus A_2\}$ in the particular case of $n = 4$ and is known as $A_{5,34}$.

Of some interest is the existence of a specific value of the parameter, m, for which there is additional symmetry. In the case of $n = 3$ (Kummer–Schwarz) there are $6 = 3+3$ symmetries and the unexpected symmetry harks back to the exceptional noncartan symmetries of $y'' = 0$. For greater values of n the number of symmetries appears to be $n+2$ which fits into the numerical scheme for linear equations. However, the structure of the algebra is different. The number of solution symmetries is $n-2$ and not n. Naturally homogeneity is preserved. What is additional is that the equation continues to have the $sl(2,\ R)$ subalgebra. Furthermore the coefficients of that subalgebra correspond to a linear equation of order $n-2$ and not n.[2] The algebra is $(n-2)A_1 \oplus_s \{A_1 \oplus sl(2, R)\}$.

[2] For a linear equation of order n the three elements are usually written as ∂_x, $x\partial_x + \frac{(n-1)}{2}y\partial_y$ and $x^2\partial_x + (n-1)xy\partial_y$.

Table 1 The number of Lie point symmetries for $n = 3,\ 4,\ 5,\ 6$ and 7

n	m	$i\Gamma$	Symmetries
3	3 or 0	7	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + y\partial_y,\ x^2\partial_x + 2xy\partial_y$
	$\frac{3}{2}$	6	$\partial_y,\ y\partial_y,\ y^2\partial_y,\ \partial_x,\ x\partial_x,\ x^2\partial_x$
	else	4	$\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x$
4	0	8	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{3}{2}y\partial_y,\ x^2\partial_x + 3xy\partial_y$
	$\frac{4}{3}$	6	$\partial_y,\ x\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{1}{2}y\partial_y,\ x^2\partial_x + xy\partial_y$
	else	5	$\partial_y,\ x\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{1}{2}y\partial_y$
5	0	9	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ x^4\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + 2y\partial_y,\ x^2\partial_x + 4xy\partial_y$
	$\frac{5}{4}$	7	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + y\partial_y,\ x^2\partial_x + 2xy\partial_y$
	else	6	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + y\partial_y$
6	0	10	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ x^4\partial_y,\ x^5\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{5}{2}y\partial_y,\ x^2\partial_x + 5xy\partial_y$
	$\frac{6}{5}$	8	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{3}{2}y\partial_y,\ x^2\partial_x + 3xy\partial_y$
	else	7	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + \frac{3}{2}y\partial_y$
7	0	11	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ x^4\partial_y,\ x^5\partial_y,\ x^6\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + 3y\partial_y,\ x^2\partial_x + 6xy\partial_y$
	$\frac{7}{6}$	9	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ x^4\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + 2y\partial_y,\ x^2\partial_x + 4xy\partial_y$
	else	8	$\partial_y,\ x\partial_y,\ x^2\partial_y,\ x^3\partial_y,\ x^4\partial_y,\ y\partial_y,\ \partial_x,\ x\partial_x + 2y\partial_y$

The number of symmetries for each value of n is determined by the value of m. The first column contains the value of n, the second the value of m, the third the number of symmetries and the fourth the actual symmetries

These observations naturally lead to the following conjectures for various values of m.

Conjecture In general, symmetries of an nth-order equation of the type (3) are given by

m = else	$\partial_y, x\partial_y, \ldots x^{(n-3)}\partial_y$ $y\partial_y$ $\partial_x, x\partial_x$
$m = \frac{n}{n-1}$	$\partial_y, x\partial_y, \ldots x^{(n-3)}\partial_y$ $y\partial_y$ $\partial_x, x\partial_x + \frac{n-3}{2}y\partial_y + x^2\partial_x + (n-3)xy\partial_y$
m=0	$\partial_y, x\partial_y, \ldots x^{(n-1)}\partial_y$ $y\partial_y$ $\partial_x, x\partial_x + \frac{n-1}{2}y\partial_y + x^2\partial_x + (n-1)xy\partial_y$

Equation (3) has $n + 1$, $n + 2$ and $n + 4$ point symmetries corresponding to m =else , $m = n/(n-1)$ and $m = 0$, respectively. The corresponding algebras are $(n-2)A_1 \oplus_s \{A_1 \oplus A_2\},\ (n-2)A_1 \oplus_s \{A_1 \oplus sl(2, R)\}$ and $nA_1 \oplus_s \{A_1 \oplus sl(2, R)\}$.

Proofs of the Conjectures: We recall that the general equation is

$$\Omega : y^{(n-2)}y^n - my^{(n-1)^2} = 0. \tag{6}$$

The linearised symmetry condition is $X^{(n)}\Omega = 0$ when $\Omega = 0$, *ie*,

$$\eta^{(n-2)}y^{(n)} - 2m\eta^{(n-1)}y^{(n-1)} + \eta^{(n)}y^{(n-2)} = 0. \tag{7}$$

Replace $y^{(n)}$ by $\frac{my^{(n-1)^2}}{y^{(n-2)}}$ in (7) to obtain

$$m\eta^{(n-2)}y^{(n-1)^2} - 2m\eta^{(n-1)}y^{(n-1)}y^{(n-2)} + \eta^{(n)}y^{(n-2)^2} = 0. \tag{8}$$

When we use the formula $\eta^{(n)} = D^n\eta - \sum_{j=1}^{n}\binom{n}{j}y^{(n+1-j)}D^j\xi$, we can rewrite (8) as

$$\begin{aligned} &my^{(n-1)^2}D^{n-2}\eta - my^{(n-1)^2}\sum_{j=1}^{n-2}\binom{n-2}{j}y^{(n-1-j)}D^j\xi - 2my^{(n-1)}y^{(n-2)}D^{n-1}\eta \\ &+2my^{(n-1)}y^{(n-2)}\sum_{j=1}^{n-1}\binom{n-1}{j}y^{(n-j)}D^j\xi + y^{(n-2)^2}D^n\eta \\ &-y^{(n-2)^2}\sum_{j=1}^{n}\binom{n}{j}y^{(n+1-j)}D^j\xi = 0. \end{aligned} \tag{9}$$

In Eq. (9) the $D^n\eta$ term has $y^{(n)}$ which we can replace by using Eq. (3). Comparing the coefficients of $y'y^{(n-2)}y^{(n-1)^2}$ in Eq. (9) we get

$$\xi_y = 0, \tag{10}$$

that is, $\xi = a(x)$.

We rewrite Eq. (9) to obtain

$$\begin{aligned} &my^{(n-1)^2}D^{n-2}\eta - my^{(n-1)^2}\sum_{j=1}^{n-2}\binom{n-2}{j}y^{(n-1-j)}a^{(j)} \\ &-2my^{(n-1)}y^{(n-2)}D^{n-1}\eta + 2my^{(n-1)}y^{(n-2)}\sum_{j=1}^{n-1}\binom{n-1}{j}y^{(n-j)}a^{(j)} \\ &+y^{(n-2)^2}D^n\eta - y^{(n-2)^2}\sum_{j=1}^{n}\binom{n}{j}y^{(n+1-j)}a^{(j)} = 0. \end{aligned} \tag{11}$$

By comparison of the coefficients of $y'y^{(n-1)}y^{(n-2)^2}$ in Eq. (11) we see that

$$\eta_{yy} = 0, \tag{12}$$

that is, $\eta = b(x) + yc(x)$.

When we use Leibnitz' rule for differentiating a product, we compute $D^n\eta$ as

$$D^n\eta = b^n(x) + \sum_{k=0}^{n}\binom{n}{k}y^{(k)}c^{(n-k)}. \tag{13}$$

On substitution of Eq. (13) into Eq. (11) we have

$$
\begin{aligned}
& m y^{(n-1)^2} b^{(n-2)} + m y^{(n-1)^2} \sum_{k=0}^{n-2} \binom{n-2}{k} y^{(k)} c^{(n-2-k)} \\
& - m y^{(n-1)^2} \sum_{j=1}^{n-2} \binom{n-2}{j} y^{(n-1-j)} a^{(j)} - 2m y^{(n-1)} y^{(n-2)} b^{(n-1)} \\
& - 2m y^{(n-1)} y^{(n-2)} \sum_{k=0}^{n-1} \binom{n-1}{k} y^{(k)} c^{(n-1-k)} \\
& + 2m y^{(n-1)} y^{(n-2)} \sum_{j=1}^{n-1} \binom{n-1}{j} y^{(n-j)} a^{(j)} + y^{(n-2)^2} b^{(n)} \\
& + y^{(n-2)^2} \sum_{k=0}^{n} \binom{n}{k} y^{(k)} c^{(n-k)} - y^{(n-2)^2} \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)} a^{(j)} = 0.
\end{aligned}
\tag{14}
$$

By comparison of the coefficients of $y^{(n-1)^2}$ in Eq. (14) we get

$$
b^{(n-2)} + (n-2) y c^{(n-2)} = 0, \tag{15}
$$

that is,

$$
\begin{aligned}
b^{(n-2)} &= 0 \\
c^{(n-2)} &= 0.
\end{aligned}
\tag{16}
$$

When we compare the coefficients of $y^{(n-1)^2} y^{(n-4)}$ and $y^{(n-1)} y^{(n-2)} y^{(n-3)}$ in Eq. (14) we obtain

$$
\begin{aligned}
c^{(2)} - \frac{n-4}{3} a^{(3)} &= 0 \\
c^{(2)} - \frac{n-3}{3} a^{(3)} &= 0.
\end{aligned}
\tag{17}
$$

The solution of (17) are

$$
c^{(2)} = 0 \quad \text{and} \quad a^{(3)} = 0. \tag{18}
$$

We use Eqs. (16) and (18) to remove all summations in Eq. (14) and the $y^{(n-1)^2} y^{(n-2)}$ terms vanish. Finally we have

$$
\begin{aligned}
& m(n-2) y^{(n-1)^2} y^{(n-3)} c^{(1)} - m \binom{n-2}{2} y^{(n-1)^2} y^{(n-3)} a^{(2)} \\
& - 2m(n-1) y^{(n-1)} y^{(n-2)^2} c^{(1)} + m(n-1)(n-2) y^{(n-1)} y^{(n-2)^2} a^{(2)} \\
& + n y^{(n-1)} y^{(n-2)^2} c^{(1)} - \frac{n(n-1)}{2} y^{(n-1)} y^{(n-2)^2} = 0.
\end{aligned}
\tag{19}
$$

When we compare the coefficients of $y^{(n-1)^2} y^{(n-3)}$ in Eq. (19), we obtain

$$
2c^{(1)} - (n-3) a^{(2)} = 0. \tag{20}
$$

From a comparison of the coefficients of $y^{(n-1)} y^{(n-2)^2}$ in Eq. (19) we obtain

$$
(n - 2m(n-1)) c^{(1)} + (m(n-1)(n-2) - \frac{n(n-1)}{2}) a^{(2)} = 0. \tag{21}
$$

We solve Eqs. (20) and (21) and obtain $c_2 = a_3 = 0$. Finally we have

$$\xi = a_1 + a_2 x \quad \text{and} \tag{22}$$

$$\eta = b_1 + b_2 x + \cdots + b_{n-2} x^{n-3} + c_1 y. \tag{23}$$

□

The Case $m = \frac{n}{n-1}$

In the case that $m = \frac{n}{n-1}$ Eqs. (20) and (21) are same and we obtain the relation

$$c_2 = (n-3)a_3. \tag{24}$$

From this relation we have one additional symmetry

$$x^2 \partial_x + (n-3)xy\partial_y. \tag{25}$$

The Case $m = 0$

The equation is

$$b^{(n)} + \sum_{k=0}^{n} \binom{n}{k} y^{(k)} c^{(n-k)} - \sum_{j=1}^{n} \binom{n}{j} y^{(n+1-j)} a^{(j)} = 0. \tag{26}$$

We collect the constant and y coefficients in Eq. (26) and find that

$$b^{(n)}(x) = 0\,, \; c^{(n)}(x) = 0. \tag{27}$$

We rewrite Eq. (26) as

$$\sum_{k=1}^{n-1} \binom{n}{k} y^{(k)} c^{(n-k)} - \sum_{j=2}^{n} \binom{n}{j} y^{(n+1-j)} a^{(j)} = 0 \tag{28}$$

and collect the coefficients of $y^{(n-1)}$ and $y^{(n-2)}$ in Eq. (28) to obtain

$$2c^{(1)} - (n-1)a^{(2)} = 0 \tag{29}$$

$$3c^{(2)} - (n-2)a^{(3)} = 0. \tag{30}$$

From Eq. (29) $c = \frac{n-1}{2} a^{(1)}$ and we substitute this into Eq. (30) to obtain $a^{(3)} = 0$, that is,

$$a = a_1 + a_2 x + a_3 x^2. \tag{31}$$

We have $a^{(3)} = 0$. If we substitute this into Eq. (28) and compare the coefficients of derivative of y up to $n-2$, we see that

$$c^{(2)} = 0\,, \; c^{(3)} = 0, \ldots, c^{(n)} = 0, \tag{32}$$

that is,

$$c = c_1 + c_2 x. \tag{33}$$

Substitute (31) and (33) into Eq. (29) to obtain the relation

$$c_2 = (n-1)a_3. \tag{34}$$

The coefficient functions of the symmetries of the case $m = 0$ are

$$\xi = a_1 + a_2 x + a_3 x^2 \quad \text{and}$$

$$\eta = b_1 + b_2 x + \cdots + b_n x^{n-1} + c_1 y + a_3(n-1)xy.$$

Singularity Analysis

We examine the specific class of equations for the value of $m = \frac{n}{n-1}$ introduced above in terms of singularity analysis. We follow the general method as outlined in [17,18] with the modification for negative nongeneric resonances introduced by Andriopoulos et al. [2]. We illustrate the method on the fifth-order equation,

$$y'''y''''' - \tfrac{5}{4}y''''^2 = 0. \tag{35}$$

To determine the leading-order behaviour we set $y = \alpha\chi^p$, where $\chi = x - x_0$ and x_0 is the location of the putative singularity. We obtain

$$\alpha^2 p^2(p-1)^2(p-2)^2(p-3)(p-4)\chi^{2p-8} - \frac{5}{4}\alpha^2 p^2(p-1)^2(p-2)^2(p-3)^2\chi^{2p-8}$$

which is zero if $(p-4) = 5/4(p-3)$, *ie*, $p = -1$. Note that the coefficient of the leading-order term is arbitrary.

To establish the terms at which the remaining constants of integration occur in the Laurent Expansion we make the substitution

$$y = \alpha\chi^{-1} + m\chi^{-1+s},$$

where the various values at which s may take are determined by the coefficient of m being zero and so arbitrary. The coefficient of m is a fifth-order polynomial the roots of which are

$$s = -1,\ 0,\ 1,\ 2,\ 3.$$

Consistency is automatically satisfied as both terms in the equation are dominant.

Conjecture The exponent of the leading-order term and the resonances of the nth member of the class of equations,

$$(n-1)y^{(n-2)}y^{(n)} - ny^{(n-1)^2} = 0, \quad n \in N > 1, \tag{36}$$

are $p = -1$ and $s = -1,\ 0,\ 1,\ 2, \ldots, n-2$.

Proof We substitute $y = \alpha\chi^p$, where $\chi = x - x_0$ and x_0 is the location of the putative singularity. We remove a common factor $p^2(p-1)^2 \cdots (p-n+3)^2(p-n+2)$. The values of p removed are all positive and so of no relevance to the singularity analysis. The remaining terms are

$$(n-1)(p-n+1) - n(p-n+2)$$

which, when put equal to zero, give the singularity to be $p = -1$.

We write $y = \alpha\chi^{-1} + \mu\chi^{-1+s}$ and substitute into (36). We remove the common factors $\chi^{-2}(-1)(-2)\cdots(-n+2)(-1+s)(-2+s)\cdots(-n+2+s)$. This immediately gives the resonances, $s = 1,\ 2,\ \ldots,\ (n-2)$, which are all positive. The remaining terms are

$$(n-1)(-n+1)(-n) + (n-1)(-n+1+s)(-n+s) - 2n(-n+1)(-n+1+s).$$

When this equated to zero, we obtain the two additional resonances $s = -1,\ 0.$ □

Apart from the generic resonance of -1 all of the resonances are nonnegative integers, *ie*, the Laurent Expansion is a Right Painlevé Series [7].

Discussion

We commenced our study with the Kummer–Schwarz Equation, (1), which is an interesting equation in its own right if for nothing else that it has six Lie point symmetries in contrast to the possible four, five or seven for a linear third-order equation. When we looked at equations of higher (also lower in one instance) order, we discerned several features which made the class of equations we have presented interesting.

The first was that the Kummer–Schwarz Equation is not generic. In [12] it was reported that linear equations of the nth order had three possibilities after one left the second order. A linear equation could have $n+1$, $n+2$ or $n+4$ Lie point symmetries. It was not possible for a linear equation to have $n+3$ point symmetries. Consequently the Kummer–Schwarz Equation, apart from the possession of the maximal number of contact symmetries, was unusual in the set of third-order equations. In a sense the Kummer–Schwarz Equation is an intermediary between the second-order equation, which has the maximal eight Lie point symmetries to be had for a second-order equation, and the higher-order equations for which the number of symmetries is $n-2$ for the specific value of m indicated for that order. The quasipersistence of a remnant of a noncartan does strike one as something peculiar.

For the higher elements of this class of equations the pattern of the Lie point symmetries is quite clear.

For the specific value of $m = n/(n-1)$ the solution of the nth-order equation is simple. One can write

$$(n-1)y^{(n-2)}y^{(n)} - n{y^{(n-1)}}^2 = 0$$

as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{{y^{(n-2)}}^{1/(n-1)}}\right) = 0$$

in which a multiplicative constant has been omitted. This elementary differential equation is easily integrated to give the solution as

$$y(x) = \frac{(n-1)^{n-1}}{(n-2)!c_1^{n-2}(c_1x+c_2)} + \sum_{i=3}^{n} c_i x^{i-3},$$

where the c_i are constants of integration.

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