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Conjecturing An Effective Problem-Solving Heuristic Instructional Approach For Circle Geometry

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Abstract- This paper reports on an investigation into finding an effective contemporary problem-solving instructional approach for circle geometry. The participants were high school learners in a South African school. This proposed instructional approach-the IPAC model, adopted four elements of this new approach, namely, the infusion approach, Polya's approach, and APOS theory in a collaborative learning classroom. This study followed a mixed-method research design. The quantitative data was analysed by implementing descriptive statistics and inferential statistics, while content analysis was performed on data extracted from qualitative questionnaires. The research findings that emanated from this study were the following: the validity, practicality and effectiveness of the designed model were established. Based on these research findings, the researcher recommended among others: (1) the designed model, should be implemented for teaching and learning of mathematics in general and circle geometry in particular, in South African schools; (2) teaching of thinking skills and teaching effective problem-solving instructional approaches should be prioritized in mathematics classrooms in South Africa.

I. INTRODUCTION

"Umalusi has observed a worrying trend in mathematics, where the subject does not seem to be progressing in tandem with other cognate subjects in terms of learner performance. Mathematics is not showing any signs of improvement, hence, Mathematics needs to be taught differently if pupils' marks are going to improve. Mathematics teachers 'must teach differently'……..'' (Volmink, 2020, p.2).

Circle geometry is an integral content under Euclidean geometry in the mathematics curriculum in South Africa. It needs to be mentioned that Euclidean geometry covers about 50 marks out of the total marks' allocation of 150, for the NSC Paper 2 mathematics examinations. This represents more than 33% of the total mark allocation for Paper 2. The low marks obtained by learners in Euclidean Geometry will inevitably affect their performance in mathematics. As mathematics teachers, we always spend substantial time, motivating and encouraging learners. Most importantly, educating them about its relevance, apart from the fact that Euclidean Geometry carries a significant part of the mark allocation in examinations, Department of Basic Education (DoBE, 2018; Abakah, 2019).

The researchers purposefully perused into learner-centred approaches. To this end, it's more advanced form i.e Thinking-Based Learning (TBL) approach, was ideal for this study (Swartz, Costa, Beyer, Reagan & Kallick, 2010). According to (Swartz et al, 2010), the goals of Thinking-Based-Learning are in three facets: (1) during schooling era, students thinking will become better; (2) students will become better with content learning; (3) after schooling era, students use of good thinking will not end, but rather, they will carry on with it by applying it in their every-day lives and professional work. This further reiterates the need to inculcate teaching of thinking skills in South Africa's mathematics classrooms.

Teaching of thinking was considered ideal for this study, based on the premise that 'learning to think' and 'thinking to learn' promotes deep and lasting learning (Ritchhart & Perkins, 2004). As opined by (Polya, 1945), "problem-solving skills are not an inborn talent but rather, they need to be learnt and practiced". Thinking skills is a vital component of effective problem-solving, especially in mathematics, thus, cannot be dissociated from solving problems in mathematics classrooms. This presupposes that teaching thinking skills is as important as teaching content knowledge. Hence, they need to be learnt and adequately practiced by learners. Teaching thinking skills has become relevant due to inability of learners to apply appropriate reasoning and thinking skills in examinations. This is made much evident as a greater percentage of learners normally abandon questions which demand higher order reasoning and advanced mathematical thinking. The few learners who attempt to solve those supposedly difficult questions end up giving irrelevant responses to those questions. In light of the above, as the researchers, we have realised that thinking skills are important problem-solving skills. Hence, learners need to learn and sufficiently practice them, just as other problem-solving skills (Diagnostic report, 2020).

In teaching thinking skills, the researchers adopted and inculcated the infusion approach (Swartz, 1992). That is teaching thinking skills, along with content instructions in this proposed instructional approach which focused on: 'teaching of thinking' and 'teaching for thinking' (Zulkpli, Abdullah, Kohar & Ibrahim, 2017). (Laborde, 2005) is of the view that students are unable to use theoretical statements in deductive reasoning. (Groth, 2005) also informs us that students encounter difficulties related to measurement, deductive proofs and linking chains of reasoning, when solving geometry problems. If one also looks at the diagnostic report by the Department of Basic Education (DOBE, 2017) "... many learners performed poorly in questions that demanded analytical, evaluative and problem-solving skills and candidates were severely disadvantaged by their lack of these cardinal skills".

The above inform that students substantially lack thinking skills, which is one of the competence mastery skills expected for solving circle geometry problems effectively. The researchers have realised that lack of thinking skills is worrisome and a serious cause of concern. This deprives students, the ability to be able to solve circle geometry problems well. Hence, thinking skills need to be taught (Swartz & Reagan, 1998). In light of the above, the infusion approach was introduced in an effort to teach students thinking skills in the circle geometry classroom. This required the prescribed content material to be restructured so that the teaching of thinking skills can be appropriately integrated into the conventional instruction (Aizikovitsh & Amita, 2010).

(Schweiger, 2003) posits that, in as much as there is increasing advocacy for teaching problem-solving, yet it's precise meaning and how it can be taught, remains a challenge to mathematics instructors. In the researchers bid to teach students how to understand a problem and how to effectively approach a problem, the researchers adopted Polya's problem-solving model: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. Polya's model guided the classroom discussion phase of the Activities, Classroom, Discussion and Exercises (ACE) teaching cycle, associated with the APOS theory. To effectively guide the design and implementation of this proposed instructional approach, the researchers employed the APOS theory (ACE teaching cycle). The APOS theory was as well, implemented to monitor learners' mental constructions.

Under the said proposed instructional approach, lessons were carried out in a collaborative learning environment setting, which has been proven to promote improved higher order learning abilities (Brijlall, 2015). This proposed problem-solving instructional approach was tentatively, known as, "IPAC mathematics problem-solving instructional model''. It was simply, labelled as the "IPAC model". This is an acronym for the four elements of this new approach, namely, the infusion approach, Polya's model, and APOS theory in a collaborative learning classroom.

The main elements which the researchers incorporated in this problem-solving instructional model, may be viewed as belonging to two distinct facets: one pertaining to teaching (the pedagogical dimension) and one pertaining to learning (the cognitive development dimension). The ACE teaching approach associated with APOS theory, Polya's problem-solving regime and Infusion as a technique of the cognitive development of thinking skills may constitute the pedagogical dimension. While the sub-constructs propelling this model on-interiorization and encapsulation or reification towards the mathematical object, circle geometry, may constitute the learning dimension of the proposed IPAC model.

To this end, the researchers; aims to foster a rationale for the use of a problem-solving instructional approach as a reliable instructional tool in the mathematics classroom. With this aim in mind, the following critical research question was formulated: how can the proposed instructional approach to be used as a problem-solving heuristic be developed and implemented in the circle geometry classroom?

II. METHOD OF RESEARCH

A mixed-method research design approach was employed in this study. The researchers realised that collecting qualitative and quantitative data about the same time would be necessary. This would give more evidence and valid data to be able to answer the research questions effectively, hence, its inculcation as the research approach for this study.

III.RESEARCH DESIGN

After detailed consideration of a bulk of research designs, the researchers, with reference to literature and from expert guidance, settled on Educational Design Research (EDR), simply termed as the Design Research, as the research design for this research study. This was proposed by Freudenthal in 1991. The researchers opted for EDR due to the unique nature of this research study - to develop knowledge and solutions. This study is about development and implementation of a proposed problem-solving instructional approach, which is to be used as an intervention, to improve the teaching and learning of circle geometry concepts.

IV. RESEARCH FRAMEWORK

Study participants

Grade 11 mathematics learners from a high school in South Africa were targeted to serve as participants for this. They were divided into two groups: one group served as the control group, while the other group served as the experimental group. Each group attended mathematics lessons at different times. The researchers served as teachers for both groups: the control group were taught by the traditional approach, while the experimental group were taught by the intervention approach. The two classes were learners from the same school who have being learning under the same teacher and other learning conditions. Hence, the two groups were homogeneous or shared the same/similar characteristics enough to be used for the study. 11A had 36 learners, comprising of 21 females and 15 males, while 11B had 32 learners, comprising of 18 females and 14 males. Learners in the two classes were between the ages of 15-18 years, from different ethnic and social backgrounds.

Development of the proposed IPAC model

Three distinct theories/approaches inspired the design of this proposed instructional approach: the APOS theory, Polya's problem-solving instructional approach, and the infusion approach. Also, lessons were conducted in a collaborative classroom setting. The main purpose the APOS theory served is to guide the design and implementation of the proposed problem-solving instructional approach. It was also used to monitor learners' mental constructions. Polya's problem-solving instructional approach was only employed to guide the classroom discussion phase of the ACE teaching cycle. However, the infusion approach was implemented at the second stage of Polya's problem-solving instructional approach (devising a plan stage), characterized by brainstorming, problem-solving and decision making (Swartz & Reagan, 1998). In addition to the above, the infusion approach was adopted so that relevant tasks that will incorporate thinking skills into circle geometry as a content field will be selected and incorporated appropriately. According to the National Council of Teachers of Mathematics (NCTM, 2000), teachers can ask questions (problem posing) which will demand students to be critical thinkers, hence, teachers need to select relevant and meaningful tasks during lessons. Problem posing, problem-solving, and conjecturing are three important mathematical activities, according to (NCTM, 2000). Also, problem-based instructions provide opportunities for students to develop their reasoning, sense-making skills and meaning (NCTM, 2009).

In addition to the above, (Zulkpli, Abdullah, Kohar & Ibrahim, 2017; Stylianides, 2007; Caram & Davis, 2005), informs that mathematics teachers need to use a variety of teaching methods such as questioning skills and strategies. These will allow them to ask questions that challenge students' cognitive ability. This will also guide students to apply different thinking strategies such as generalising, applying, analogising, explaining, finding evidence, examples and representing the subject in a new way. This is because questioning strategies can intrigue curiosity, stimulate interest and intrinsic motivation for students to obtain new information (Caram & Davis, 2005). In light of the above, the infusion approach was implemented during the "Discussion phase", "Activities phase" and the "Exercise phase" of the ACE teaching cycle. It was as well implemented when formulating questions that constituted the standardized tests. This is because it is only the "Discussion phase", "Activities phase", the "Exercise phase" and constituting standardized tests, which will require questions or problems/activities /tasks to be selected by the researchers. As informed by (Mudrikah, 2016), problem- based learning is appropriate to be used to improve students' high order mathematical thinking ability since it can propel the reflective abstraction related mental actions, mental processes, mental objects and schemas in students.

Explication and implementation of the proposed IPAC model

Sequentially, the following procedures as elaborated below, were adhered to, on the experimental group:

(1) Generation of genetic decomposition (GD) for circle geometry concept(s) from the theoretical analysis of the circle geometry concept(s). Circle geometry lessons were based on the mental constructions that learners require at that stage of genetic decomposition (Tziritas, 2011). This was sub-divided into four mental construction lessons: GD1- Action stage of circle geometry mental construction lesson, GD2- Process stage of circle geometry mental construction lesson, GD3- Object stage of circle geometry mental construction lesson, and GD4 - Schema stage of circle geometry mental construction lesson. The details of each of the four stages are elaborated below:

GD1- Action stage of circle geometry mental construction lesson

Learners are expected to be able to solve problems by following detailed step by step knowledge procedures. This may require specific teaching, and the need to perform each step clearly. Straight recall and use of circle geometry theorems and geometric language, as well as conceptual knowledge and understanding of geometric concepts are required at this stage. Questions which are relevant to this stage will be administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach). As mentioned earlier, developing students thinking skills demands students to be exposed to unfamiliar questions and tasks by depending on their relevant previous knowledge. In view of the above, this GD1 lesson, will serve as the prior knowledge that needs to be established by students, which will guide them in the development of higher thinking skills (GD2, GD3 & GD4 lessons), as elaborated below. (Tziritas, 2011; Maharaj, 2010; Swartz & Reagan, 1998; CAPs, 2012; King, Goodson & Rohani, 2013).

GD2- Process stage of circle geometry mental construction lesson

Learners at the process level are expected to be able to reflect on the action process, describe, or even reverse the steps of previously learned objects without performing those steps. Learners are expected to be able to appropriately distinguish between the different geometry theorems and to know when and how to use each theorem in each problem-solving situation. Learners are also expected to be able to prove and perform simple applications of the circle geometry theorems they learnt during the action stage. As learners can continuously repeat and reflect on an action, it may be interiorised into a mental process. Questions which are relevant to this stage will be administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach). (Tziritas, 2011; Swartz & Reagan, 1998; CAPs, 2012).

GD3- Object stage of circle geometry mental construction lesson

Learners at the object level are expected to be able to reflect on a particular set of processes until they are able to perform encapsulations on the mathematical concept. Learners at this stage can be said to have encapsulated the process into a cognitive object. At this stage, higher order reasoning is required since there is no obvious route to the solution to the problem which could involve making significant connections between different geometric concepts which demands adequate conceptual understanding and application of geometric concepts. Also, questions which are relevant to this stage will be administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach). (Tziritas, 2011; Swartz & Reagan, 1998; CAPs, 2012).

GD4- Schema stage of circle geometry mental construction lesson

At this level: Actions, processes and Objects of a mathematical concept are interconnected in the learner's mind to construct schemas. Learners are expected to be able to organise and link these stages together to form a coherent framework (schema). This stage is characterized by non-routine problems which requires higher order reasoning and processes. Breaking the problem into its constituent parts to reach the solution to the problem may be performed at this stage. The schema level of thinking more than any other, requires a culmination of the infusion approach – in fact, schema level cannot be reached without active infusion of thinking skills into the content area, in this case, circle geometry. Hence, the teacher him/herself modelling reasoning (in a circle geometry problem) at the schema level, would be crucial within the infusion process. Lastly, questions which are relevant to this stage will be administered to learners, during the discussion phase, activities phase and exercise phase to incorporate thinking skills into the conceptual understanding of circle geometry concepts (infusion approach). (Tziritas, 2011; Swartz & Reagan, 1998; CAPs, 2012).

(2) Class discussion phase of the ACE teaching cycle will be guided by Polya's problem-solving instructional approach: understanding the problem, devising a plan, carrying out the plan, and reviewing the steps. It will also be guided by the infusion approach. Questioning skills that can improve students thinking skills, as advocated by the infusion approach will be ensured. The procedure for the class discussion phase is elaborated as follows (adapted from Abakah,2019):

(a) The teacher gives a leading question (in the case of a new concept) or write a problem to be solved on the board (in the case of continuation of the previous concept).

(b) The study participants start to discuss the solution in view of Polya's problem-solving approach steps enumerated earlier, characterized by problem posing, problem-solving, and conjecturing. What learners are expected to do at each step is delineated below:

(i) Step 1. Understanding the problem
Learners are expected to carefully read and understand the problem to be solved, to paraphrase the problem in their own words, if necessary, to emphasize what they understood, and to determine what the problem asks them to solve, that is, to determine the unknown.

(ii) Step 2. Devising a plan (thinking stage).
Learners are mandated to think rigorously and endlessly, until a reasonable solution to the question/problem is reached.

(iii) Step 3. Carrying out the plan.
Learners are expected to be able to implement the strategy/thought they devised in the previous step by performing necessary actions or computations.

(iv) Step 4. Looking back
Learners are taught to check the validity of the final solution they found. Most importantly, they are asked to interpret the result they found and to determine whether the solution makes sense and is reasonable in the context of the problem. (i-iv Adapted from: Valles & Wickramasingh, 2015).

(c) The teacher goes round each group to moderate or correct the groups' discussions.

(d) The teacher stops the discussion and allow the study participants to present their solutions and allow other groups to criticise/support each other's solutions.

(e) The teacher finalised the solution by accepting or correcting the solution proposed by the study participants. He then gives more detailed explanation on the problem(s) before introducing another problem to be solved to the study participants.

Thinking skills, processes and dispositions to be taught under each genetic decomposition (GD) lesson

The development of students' thinking calls for cultivating their skills, processes, and dispositions concerning better thinking (Swartz & Perkins, 1998). In view of this, the researchers found it relevant to present how each GD lesson was aligned to the thinking skills, processes, and dispositions to be taught in the circle geometry classroom (see Table 1). They are sequentially presented in the table below. However, how thinking skills, together with their corresponding processes, and dispositions were implemented in this study, is elaborated under details of how lessons were conducted on the experimental group.

Table 1: Thinking skills, processes and dispositions to be taught under each GD lesson

GD lesson	Thinking skills to be taught	Process which enhances thinking skills	Disposition to promote lesson	Questioning approach to be used for lesson
GD1 **(Introductory lesson)**	Understanding and retention of ideas	Decision making	Making thinking 'clear and careful'	Recalling of content knowledge by asking direct procedure questions
GD2	Generating ideas (Creative thinking)	Decision making and problem solving	Making thinking 'clear and careful'	Asking indirect procedure questions, which requires application of GD 1 knowledge
GD3	Assessing reasonability of ideas (Critical thinking).	Decision making and problem solving	Making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time.	Asking higher order questions
GD4	Blending generating of ideas in GD2, with assessing reasonability of ideas in GD 3, and engaging students in metacognitive reflections	Decision making and problem solving	Making thinking 'clear and careful', 'adventurous and broad', making thinking organized, and giving thinking time.	Asking higher order questions

(Swartz & Perkins, 1990; Swartz & Reagan, 1998)

Ethical considerations

An ethical clearance letter from the education department gave us permission from the university to conduct this research study as the research field was obtained. Also, concomitantly, permission letter from the provincial department of education, district department of education, school governing bodies (SGB) of schools in which this research was conducted, learners, parents/ guardians' consent were all obtained. Confidentiality-Anonymity was ensued by not requesting for any form of identification from the study participants. The identities of all persons who participated in this research study were precluded and they were never made known in any way.

V. DATA ANALYSIS

This research study followed a mixed-method research design, hence, the qualitative data were analysed separately, as well as the quantitative data. They were then consolidated as one. From this, more meaningful and valid conclusions were made. Firstly, content analysis was performed on data extracted from qualitative questionnaires. Secondly, descriptive statistics was used to analyse the quantitative data - relevant tables were created, graphical representations, and statistical numerical calculations (mean, mode, median, standard deviation, et cetera) which were relevant to describe activities or events at the research field were done. Thirdly, inferential statistics was as well, used to analyse the quantitative data- hypothesis test was conducted.

Data analysis of questionnaires

This section elaborated on data presentation and analysis of the questionnaires used for this study- teacher observer's questionnaire, the HOD as an observer questionnaire and participants' questionnaire. In this section, each question of each part of each questionnaire and their responses were summarised in a table for easy comprehension. Also, a scan of actual responses of teacher, HOD and participants, were presented for readers' comprehension, as delineated below.

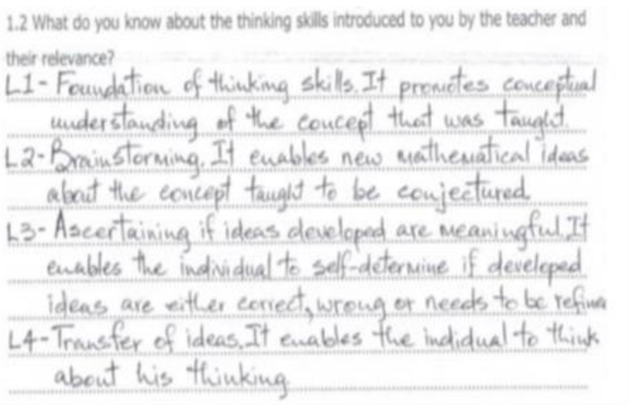
Teacher/HOD questionnaire

This questionnaire was designed to enable the researchers to know how the new instructional approach was implemented and how well lessons were understood. This questionnaire was answered by the substantive mathematics teacher and HOD for mathematics at the research field. The items of this research questionnaire were divided into four parts. Part A: how teaching of thinking skills was conducted, Part B: mode of presentation of lesson; Part C: how well you understood the lesson; and Part D: any other comments/remarks.

Analysis and discussion Of "Part A" of Teacher/HOD questionnaire

As said before, "Part A'' of this questionnaire sought to ascertain how teaching of thinking skills, sacrosanct to this study, were conducted. It was observed that both observers- teacher and HOD, unanimously gave synonymous responses in this section, which is good for data validation and replication (McMillan & Schumacher, 2014). Responses from the two observers showed that the expected thinking skills were covered: lesson one- Understanding and retention of ideas, lesson two-Generating ideas (Creative thinking), lesson three- Assessing reasonability of ideas (Critical thinking) and lesson four- Blending generating of ideas in lesson 2, with assessing reasonability of ideas in lesson 3, and engaging students in metacognitive reflections (Swartz & Reagan, 1998). Most relevantly, the relevance of these thinking skills, paramount to this study, were also stated. The responses from the two observers are presented below:

Teacher's response


1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1- Foundation of thinking skills. It promotes conceptual understanding of the concept that was taught.
L2- Brainstorming. It enables new mathematical ideas about the concept taught to be conjectured.
L3- Ascertaining if ideas developed are meaningful. It enables the individual to self-determine if developed ideas are either correct, wrong or needs to be refin
L4- Transfer of ideas. It enables the indidual to think about his thinking.

HOD's response

1.2 What do you know about the thinking skills introduced to you by the teacher and their relevance?

L1- Preliminary thinking skills to be learnt by students. It directs students to master and recall mathematical concepts taught.
L2 - Forming new mathematical knowledge. Studies are tasked to broaden their horizon, when solving problems
L3 - Judging level of corrections of ideas. Students are guided by this to be able to know if they are on the right path or on the wrong path of solving a problem
L4 - Application of ideas, concisely and logically to solve prob Students. This will guide students to be able to develop the competence of solving non-routine problems well.

It can be observed from the responses above that the thinking skills to be taught were appropriately scaffolded from the basic to the complex thinking skills. This made these thinking skills to be taught strategically to any domain of participants-the weak, the average and the strong learners. Most importantly, the role any of these four thinking skills played when solving problems were also stated for any individual to understand these thinking skills-what they stand for, what they entail and how they can be introduced to learners. Also, integral to this section was how the Teacher researchers (T-R) guided participants to solve problems using these thinking skills. The responses of the two observers to this are presented below:

Teacher's response

2.1 How did the teacher guide students on how to use thinking skills to solve problems

Learners were placed at the centre of learning. They were required to discuss and communicate among themselves they brainstorm, look for multiple ways of answering a question, consider other alternatives provided by other group members. The teacher only served as a prompter. He went to each to guide them to know if they are on the right path or wrong path on some occasions on other occasions he allows each group to present their solution on the board. This enabled other members to constructively criticize the solution provided or to support it.

HOD's response

2.1 How did the teacher guide students on how to use thinking skills to solve problems

For all lessons I observed, teaching and learning was conducted in a collaborative classroom setting - learners sitting, discussion and interacting in each other in groups of three. Thinking skills was purposefully implemented during the questioning stage of lesson, to guide learners to reach solutions to problems. After learners having understood the question, they are been nurtured to be able to brainstorm, conjecture, ideas, trial-and-error, consider more alternatives of solving the same question. They are urged to keep trying until a meaningful solution to a problem is reached.

The responses above ascertained those lessons were conducted in a collaborative classroom setting, which is apropos for nurturing learners' thinking and reasoning abilities (Brijlall, 2015). Also, participants were placed at the centre of their learning, a condition which is prioritized by the constructivists (Ekawati et al, 2019). The above confirmed that an appropriate teaching strategy (constructivist approach) and an appropriate learning environment (collaborative classroom setting) were created during lessons, which enhanced participants' thinking (King, Goodson, & Rohani, 2013). Also, from the responses, participants were encouraged to find varied ways a particular problem can be solved. In addition, participants discussed, interacted and engaged among themselves when solving problems (Brijlall, 2015). This by so doing nurtured participants to be creative thinkers (Swartz, 2012). Most integral to nurturing the thinking capacity of participants is the questioning approach implemented during lessons-challenging and higher order questions. This questioning approach proved to be a powerful tool in developing participants thinking competence. This intrigued and elicited their thinking to the required limit, which assisted in reaching desired solutions to the given problem (Mudrikah, 2016; Nafisah et al, 2011; Best, 2019).

Analysis and discussion Of "Part B" of Teacher/HOD questionnaire

This part of the questionnaire ascertained the mode of presentation of the lessons. The responses to the above by the two observers were presented below for readers' perusal:

Teacher's response

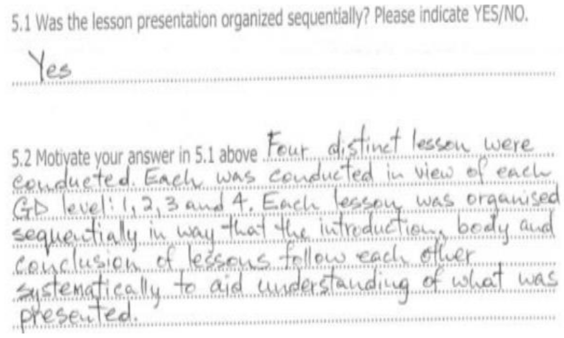
5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

YES

5.2 Motivate your answer in 5.1 above

Each lesson was orderly presented and structured, It was properly scaffolded into the introductory part, then the development of the lesson itself, then finally the Conclusion part

HOD's response



5.1 Was the lesson presentation organized sequentially? Please indicate YES/NO.

Yes

5.2 Motivate your answer in 5.1 above Four distinct lesson were conducted. Each was conducted in view of each GD level: 1, 2, 3 and 4. Each lesson was organised sequentialy in way that the introduction, body and conclusion of lessons follow each other systematically to aid understanding of what was presented.

The responses above certified how lessons were conducted during the study. Both observers attested those lessons were organized sequentially, by responding 'Yes' to sub-question 5.1 above. Observers' justifications for responding in the affirmative to sub-question 5.1, were also elaborated above. Their justification confirmed that lessons were orderly, well-structured, and properly scaffolded, to enhance participants' understanding (NCTM, 2000).

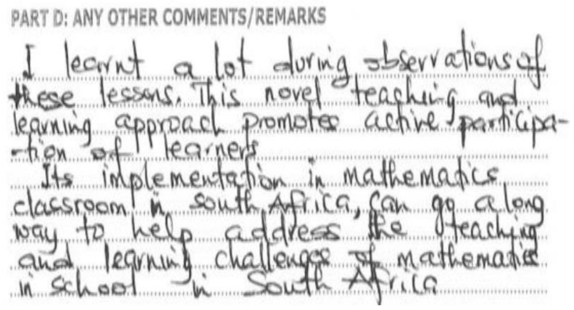
Analysis and discussion Of "Part C" of Teacher/HOD questionnaire

This part of the questionnaire sought to establish how well lessons were understood. Their responses attested that both observers inscribed 'very well understood' to question 6.1 above. Their reasons for affirmatively responding to question 6.1 above ascertained that the logical, sequential presentation and effective delivery of lessons made lessons to be very well understood (NCTM, 2000).

Analysis and discussion Of "Part D" of Teacher/HOD questionnaire

This part of the questionnaire sought to find out if the two observers had other comments/remarks with regards to the conducted lessons using the new approach. Their responses to this were captured below.

Teacher's response

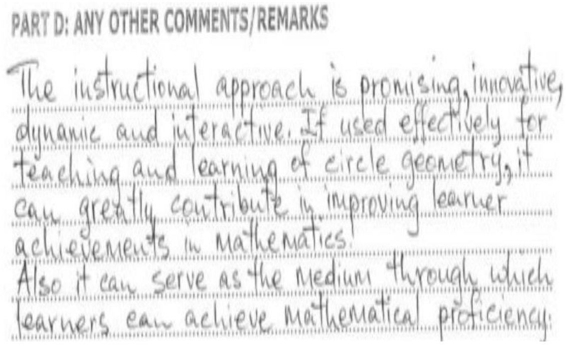


PART D: ANY OTHER COMMENTS/REMARKS

I learnt a lot during observations of these lessons. This novel teaching and learning approach promotes active participa-tion of learners.

Its implementation in mathematics classroom in South Africa, can go a long way to help address the teaching and learning challenges of mathematics in school in South Africa

HOD's response



PART D: ANY OTHER COMMENTS/REMARKS

The instructional approach is promising, innovative, dynamic and interactive. If used effectively for teaching and learning of circle geometry, it can greatly contribute in improving learner achievements in mathematics.

Also it can serve as the medium through which learners can achieve mathematical proficiency.

The unedited final comments/remarks from the two observers confirmed that the new approach is helpful in teaching circle geometry. Hence, the duo advocated for its implementation en masse in South Africa's mathematics classrooms. The duo made some interesting remarks about the new approach, which the researchers are much interested in. These remarks were: the new instructional approach is promising, innovative, dynamic, interactive, and it can be used as a medium to achieve mathematical proficiency.

Participants' questionnaire

This questionnaire was designed to measure the effects/influence the proposed problem-solving instructional approach had on learners, with regards to the teaching and learning of circle geometry. This questionnaire was answered by the study participants, individually. The items of this research questionnaire were divided into three parts. Part A: how the new instructional approach can influence participants' learning of circle geometry, Part B: how it can influence participants' problem-solving skills when solving circle geometry problems and Part C: any other comments/remarks. Participants' responses to this questionnaire were summarized on a table, before their responses were interpreted and analysed as presented below.

Analysis and discussion Of "Part A" of participants' questionnaire

This part of the questionnaire investigated how the new instructional approach influenced participants' learning of circle geometry. This was necessary as the researchers sought to know how effective the new instructional approach would be in mathematics classrooms when it is used as the medium of instruction. Some of the relevant responses from participants that attested to this are presented below for readers' examination.

Participant's response 1

Question 1

How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively

positively

Question 2

Motivate your answer in question 1 above

previously, I had difficulties with how i must go about a mathematics questio[n] This new approach has shown me the way

Participant's response 2

Question 1

How did the new instructional approach influence how you learn circle geometry? Please specify either positively or negatively

positively

Question 2

Motivate your answer in question 1 above

It gave me direction and reason for me to believe that i can also understand and solve mathematics

Participant's response 1

Question 3

From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO

yes

Question 4

Motivate your answer in question 3 above

this new approach shows me what i must do and how i must do it … a mathematics problem

Participant's response 2

Question 3

From now on, will you use the new instructional approach to learn circle geometry? Please specify either YES or NO

Yes

Question 4

Motivate your answer in question 3 above

It has assisted me greatly to learn maths meaningfully

Participant's response 1

Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO

Yes

Question 6

Motivate your answer in question 5 above

It is a good approach of learning mathematics and solving mathematic problems.

Participant's response 2

Will you recommend this new instructional approach, to any third party, for teaching and learning of circle geometry? Please specify either YES or NO

Yes

Question 6

Motivate your answer in question 5 above

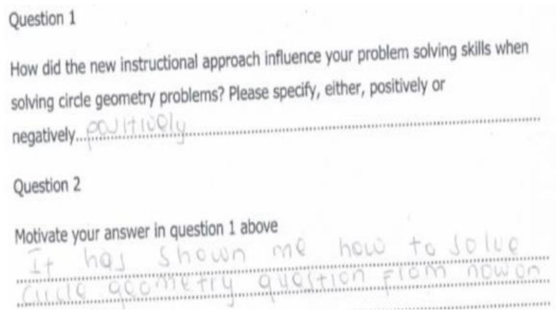
It is a good approach of learning mathematics and solving mathematic problems.

It was observed that 30 out of the 32 participants (representing 94% of participants) had indicated "Yes'' to two questions of this part of the questionnaire (questions 3 & 5). These participants indicated respectively that: from now on, they will use the new instructional approach to learn circle geometry, with the motivation that it taught them how they can learn and solve circle geometry problems well; and they Will recommend this new instructional approach, to any third party, for teaching and learning of circle geometry, with the reason that it can help others in the same ways it helped them. Also, the same 30 out of the 32 participants (representing 94% of participants) indicated for question 1 on the questionnaire that the new approach positively influenced how they learnt circle geometry, with the justification that it enhanced their confidence and it assisted them to approach circle geometry questions well. Some of the unedited responses for questions 1, 2 & 3 were presented above for verification.

Analysis and discussion Of "Part B" of participants' questionnaire

Part B of this questionnaire sought to investigate how the new approach can influence the study participants' problem-solving skills, when solving circle geometry problems. Also, 30 out of the 32 participants (representing 94% of participants) indicated on the questionnaire that an appropriate problem-solving instructional approach (Polya's approach) was implemented. These participants justified this by responding that this new approach influenced their problem-solving skills positively, it enabled them to solve circle geometry problems well, and it guided them on how they can solve problems well. Some of the unedited responses from the participants which ascertained this were presented below, for readers' perusal.

Participant's response 1


Question 1

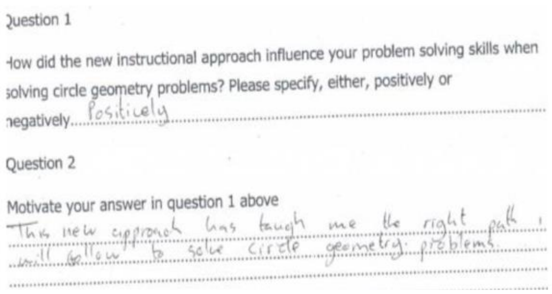
How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively...positively

Question 2

Motivate your answer in question 1 above

It has shown me how to solve circle geometry question from now on

Participant's response 2


Question 1

How did the new instructional approach influence your problem solving skills when solving circle geometry problems? Please specify, either, positively or negatively...positively

Question 2

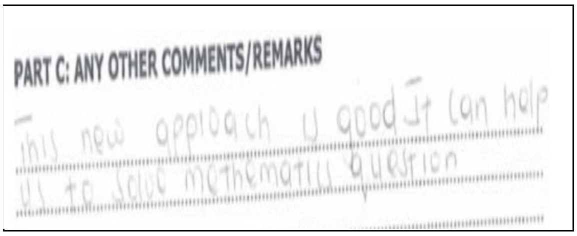
Motivate your answer in question 1 above

This new approach has taugh me the right path i will follow to solve circle geometry problems.

Analysis and discussion Of "Part C" of participants' questionnaire

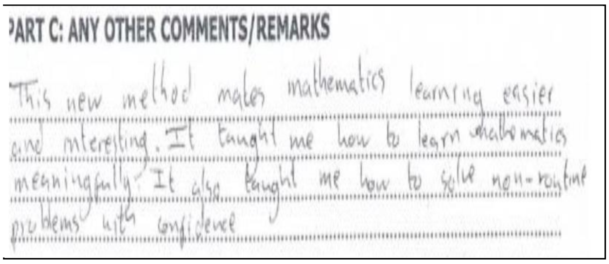
This part of the questionnaire investigated if the participants had other comments/remarks with regards to the conducted lessons which implemented the new problem-solving instructional approach. Some of their responses were captured below.

Participant's response 1


PART C: ANY OTHER COMMENTS/REMARKS

This new approach is good It can help us to solve mathematics question

Participant's response 2


PART C: ANY OTHER COMMENTS/REMARKS

This new method makes mathematics learning easier and interesting. It taught me how to learn mathematics meaningfully. It also taught me how to solve non-routine problems with confidence

The unedited final comments/remarks from two of the participants (selected at random) ascertained that the new approach is helpful in learning circle geometry. Hence, 30 out of the 32 participants (representing 94% of participants) advocated for its implementation in mathematics classrooms in South Africa. These participants indicated on the questionnaire that the new approach is good, it helped them to confidently solve non-routine circle geometry problems well, it made learning maths easier and interesting, it taught them how to learn maths meaningfully, et cetera. These remarks from the study participants ascertained that the new approach is helpful. The two participants who rejected the new approach: E_{19} and E_{25} , was not a surprise to the researchers because they were not always present during lessons, hence, they were not able to follow proceedings well and systematically. This adversely affected their composite scores-they had the least scores.

VI. DISCUSSION

Based on data analysis results the research findings that emanated from the conduct of this study were deduced. They were presented and thoroughly discussed below, in view of the research question. They are delineated below.

It was established from this study that the validity, practicality, and effectiveness of this new method of teaching and learning were justified (Nieveen, 1997; 1999). The IPAC model, is a completely new and comparatively distinct instructional approach. This was developed, tried and tested. This enabled the researchers to measure its degree of efficacy, appertaining to its validity, practicality and effectiveness (Nieveen, 1997; 1999) cited in (Fauzan, Plomp, & Gravemeijer, 2013). (Nieveen, 1997; 1999) refered to the above three measurement descriptors as "the three quality criteria check" of an intervention approach. It was also mentioned earlier that this IPAC model, entails the integration of three key theories/approaches: Infusion approach, Polya's problem-solving approach and APOS theory, in a collaborative classroom setting. This interactive, action-driven, and sophisticated problem-solving instructional approach would have been abortive without receiving expert advice and assistance with regards to its development, and most relevantly, its implementation in mathematics classrooms.

This study established that this IPAC model, is promising, in leading mathematics learners on the pathway to achieving mathematical proficiency, in conformity with its validity, practicality and effectiveness (Fauzan, Plomp, & Gravemeijer, 2013). In doing so, the researchers perused into the collected data, analysis of collected data, and results that emanated from the collected data. To this end, the information gathered informed that this IPAC model, can duly be developed and effectively implemented in mathematics classrooms, as this study sought to investigate.

VII. SUMMARY / SUGGESTIONS

This IPAC model is unprecedented. Enough evidence from this study has established that this IPAC model can be used as a tool by mathematics teachers and pedagogues to enable them to teach differently. This is because it informs about new ways and ideas of teaching and learning mathematics, which have never been considered in South Africa's context. It can as well be used as a tool by learners to develop them into effective mathematics problem solvers and good thinkers. This can lead them on the right path of achieving mathematical proficiency (DoBE, 2018). Also, adequate evidence from the conduct of this study has established that this IPAC model-mathematics teaching and learning tool, can be used as a contributory teaching and learning resource in demystifying mathematics. That is, making mathematics understandable, interactive, interesting and as a thinking-laden discipline. These, according to (Gono & Pacoy, 2021) promotes meaningful mathematics learning. Enough evidence from this study connoted that the appropriate usage of the IPAC model, will sufficiently improve learners' achievements in mathematics. This study has established that by using this IPAC model, teachers desire to teach mathematics increased. This ultimately increased learners' desire to do and learn mathematics. Furthermore, the use of this IPAC model supplemented and optimized learners' mathematics learning by incorporating active learning methods in mathematics classrooms. This ensured active participation by learners in mathematics classrooms, thereby, contributing in demystifying mathematics (Gono & Pacoy, 2021).

Additionally, the continuous implementation of this IPAC model will greatly nurture individual learners to be responsible for their own learning. Firstly, it teaches students the skill of appropriately 'planning for their learning'. Secondly, it teaches learners how they can meaningfully sequence and organize their learning. Thirdly, and most importantly, it teaches learners about metacognitive awareness and how they can individually monitor their thinking and learning. All these come together in developing students to become good learners, good thinkers and effective problem solvers (William & Maat, 2020). According to (Shannon, 2008), all these factors will cumulate into nurturing students to become effective self-directed learners.

This IPAC model will give mathematics learners and teachers, good reasons to completely discard the traditional instructional approach. This might serve as a path that leads to saying farewell to rote learning (procedural knowledge), in favour of this new instructional approach which embraces logical, creative and critical thinking, metacognition, as well as conceptual understanding of circle geometry concepts (Hirschfeld-Cotton & Nebraska, 2008).

This IPAC model may not per say serve as a complete panacea, to liberate South Africa from all her mathematics problems and difficulties. However, it may go a long way in addressing some of the challenges, especially, the methodological and pedagogical teaching and learning difficulties encountered in South Africa's mathematics classrooms. In view of the above, I strongly advocate that mathematics learners, teachers and instructors, locally and farther afield of South Africa, should make this IPAC model, their companion henceforth. Let us all unite and work towards achieving this goal.

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