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RESEARCH ARTICLE

Diffusive and dynamical radiating stars with realistic equations of state

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Abstract We model the dynamics of a spherically symmetric radiating dynamical star with three spacetime regions. The local internal atmosphere is a two-component system consisting of standard pressure-free, null radiation and an additional string fluid with energy density and nonzero pressure obeying all physically realistic energy conditions. The middle region is purely radiative which matches to a third region which is the Schwarzschild exterior. A large family of solutions to the field equations are presented for various realistic equations of state. We demonstrate that it is possible to obtain solutions via a direct integration of the second order equations resulting from the assumption of an equation of state. A comparison of our solutions with earlier well known results is undertaken and we show that all these solutions, including those of Husain, are contained in our family. We then generalise our class of solutions to higher dimensions. Finally we consider the effects of diffusive transport and transparently derive the specific equations of state for which this diffusive behaviour is possible.

Keywords Equations of state · Generalized Vaidya spacetimes · Radiating stars

1 Introduction

The geometry outside a spherically symmetric radiating star is described by the Vaidya spacetime [1] and it defines outgoing null radiation. It is written in terms of the mass of the radiating body and the Petrov–Pirani–Penrose classification of the metric is of type D [2]. The result [1] provided an advance in the modeling process and the possibility

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then arose to study the interior of radiating stars by matching the interior solution to the radiating exterior [3–7]. The complete derivation of the junction conditions for a shear-free radiating star was provided by Santos [8]. The important result that followed was that the pressure on the boundary of the radiating star should be nonzero in general, and proportional to the heat flux. It should be noted that this framework describes only the emission of pressureless null radiation (photons) into the exterior region of the dissipating star, and no other outflow of any other type of observable radiation. The effects of radiation are important in the latter stages of gravitational collapse and allows for a surrounding zone of radiation.

1.1 The problem

Though the outside geometry and matching conditions have been studied in detail, the main problem is as follows:
How do we model a realistic collapsing astrophysical star with a core null fluid and a string fluid which matches to the intermediate Vaidya spacetime enclosed by the Schwarzschild exterior?
This is a key question for a better understanding of the dynamics, thermodynamics and gravitational collapse in realistic astrophysical stars, in the context of general relativity. The class of spacetimes that are natural candidates for models of such stellar interiors are *generalised Vaidya spacetimes*. The matter field in these spacetimes have two components: A general Type I matter field (whose energy momentum tensor has a timelike and three spacelike eigenvectors), that describes null fluid matter, and also a Type II matter field (whose energy momentum tensor has double null eigenvectors) that describes null radiation and a string fluid. Such a stellar interior can then be naturally matched to an external radiating zone described by the Vaidya spacetime, and finally the radiation zone can be matched smoothly with the vacuum Schwarzschild exterior, as we explain in later sections of the paper.

1.2 Generalised Vaidya spacetimes

The generalisation of the Vaidya spacetime was given in detail by Wang and Wu [9] and includes most of the known solutions of Einstein's field equations with the additional Type II fluid. The notion that the energy momentum tensor is linear in terms of the gravitational mass for these matter fields, engenders this generalisation of the spacetime. The generalised Vaidya metric in single (exploding) null coordinates $(v, \mathsf{r}, \theta, \phi)$ is given as

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (1)$$

Here the function $m(v, \mathsf{r})$ describes the Misner-Sharp mass of the stellar interior and can be obtained via integrating the Einstein field equations with combinations of perfect fluid and null matter sources.

1.3 Earlier works

In recent times, Maharaj et al. [10] have generalised the Santos junction condition by matching the interior geometry of a manifold, containing a shear-free heat conducting fluid, to the exterior geometry described by the generalised Vaidya metric which contains the additional Type II null fluid. This result provided an even wider array of opportunities and possibilities with regards to the modeling of relativistic objects in astrophysics and cosmology. More importantly, the result provided a more general exterior region for a radiating star, which is made up of a two-fluid system: a combination of the standard null radiation as in the case of the original Santos framework, and an additional more general fluid distribution which can be taken to be another form of radiation or, perhaps more interestingly, a field of particles such as neutrinos or other exotic non-interacting matter. Another interesting feature of the generalised junction condition is the fact that the radiating fluid pressure at the boundary is not only proportional to the heat flux, but also coupled to the non-vanishing energy density of the Type II null fluid.

A significant amount of study on relativistic radiating stars has been carried out in the standard Santos framework. The junction conditions were generalised, for example, to include the effects of an electromagnetic field as well as shearing anisotropic stresses during dissipative stellar collapse by de Oliveira et al. [11] and Maharaj and Govender [12]. Analytical solutions for shear-free non-adiabatic collapse in the presence of electric charge were obtained by Pinheiro and Chan [13]. Schäfer and Goenner [14] studied a highly idealised model with constant luminosity radiating away its mass showing that an event horizon never forms. Nonlinear models for relativistic stars in the shear-free regime were found with heat flow by Misthry et al. [15] using a transformation that reduced the boundary condition to a simpler form in the conformally flat zone. Abebe et al. [16,17] utilised the Lie symmetry analysis for studies on geodesic models and radiating Euclidean stars with an equation of state. Govender et al. [18] modeled the physical behaviour at the surface of a radiating star. They investigated the effect of the exterior energy density on the temporal evolution of the radiating fluid pressure, luminosity, gravitational redshift and mass flow at the boundary of a relativistic star.

Another important notion to consider is that of gravitational collapse. Maharaj and Govender [19] studied collapse models with an internal isotropic pressure and vanishing Weyl stresses and probed the dynamical stability of the dissipating stellar fluid. They found that close to the centre, the configuration was more unstable. The thermal evolution of a radiating fluid is vital in any stellar model and Martinez [20], Herrera and Santos [21] and Govender et al. [22] studied the explicit role of relaxation and mean collision time in these frameworks. A further investigation of these ideas was carried out by Naidu et al. [23], Naidu and Govender [24] and Maharaj et al. [25] where the latter authors investigated the gravitational collapse of a radiating sphere evolving into a final static configuration described by the interior Schwarzschild solution. More recently, Mkenyeleye et al. [26] studied the gravitational collapse of the Vaidya spacetime in the context of the cosmic censorship hypothesis. Developing a general mathematical template, they showed that there exist classes of generalised Vaidya mass functions in which the collapse reaches an end state with a locally naked central singularity. It is often required that for a radiating stellar model to have a more realistic physical form, a barotropic equation of state must be imposed on the fluid distribution. In most cases the equation of state becomes a Cauchy–Euler differential equation. Several attempts have been made to model such situations. Wagh et al. [27] made use of a linear equation of state in spacetimes which are shear-free and Goswami and Joshi [28] studied the gravitational collapse of an isentropic perfect fluid distribution with a linear equation of state. An important note to make is that the notion of a Type II fluid existing in the exterior region of the radiating star has been studied in isolation without any direct connection to the interior matter conglomeration. Nonstatic spherically symmetric solutions to Einstein's field equations with a null fluid source were obtained by Husain [29] in general for such an exterior fluid with a polytropic equation of state $P = k\rho^a$. He demonstrated that for a linear equation of state ($a = 1$) and varying values of the constant k, the metrics were either asymptotically flat ($1/2 < k < 1$) or cosmological ($0 < k < 1/2$). The value $k = 1$ yielded the charged Vaidya solution. Finally, it was shown that in the long time limit, the asymptotically flat spacetimes were hairy black hole solutions. Dawood and Ghosh [30] characterised a large family of solutions to Einstein's equations representing a spherically symmetric Type II fluid, and showed that the well known dynamical black hole solutions are a particular case of this larger family. Ghosh and Dawood [31] then generalised these results to higher dimensions. An appraisal was conducted by Wang and Wu [9] where the ideas of Husain and others were extended and further classes of solution were obtained. Contained within these results are the well known monopole solution, the de Sitter and anti-de Sitter models, the charged Vaidya solution, the Husain solution and the radiating dyon solution. It should be noted that these solutions were obtained via a method of assuming a series form for the gravitational mass function in the field equations. In our study below, we will attempt to integrate the field equations directly subsequent to assuming an equation of state.

1.4 This paper

The intent of this paper is to generate solutions to the generalised Vaidya stellar interior with a string fluid and null matter for various thermodynamically realistic equations of state. It turns out that a direct integration of the resulting partial differential equations is possible in general for the linear, quadratic and polytropic equations. Our solutions for the linear cases generalise all of those obtained by Husain and others as well as the complete summary of solutions presented in [9], and are therefore the most general solutions known. For pedagogical completeness, we also further generalise all of our results in the higher dimensional generalised Vaidya spacetimes.

This paper is organised as follows: In the next section we give a complete outline of how to model an isolated spherical and physically realistic radiating astrophysical star via the generalised Vaidya geometry. In the following section we describe the generalised Vaidya spacetime in detail by analysing the Einstein field equations. We present the relevant aphorisms indicative with the geometry of the generalised Vaidya metric and make mention of the energy conditions for a physically reasonable model. A note on equations of state is presented with various cases discussed. In Sect. 4 we systematically present solutions to the Einstein field equations for the gravitational mass function by assuming several different equations of state. The succeeding Sect. 5 then deals with the higher dimensional spacetime and field equations. The solutions are summarised in detail for higher dimensions and the masses are tabulated. Finally an analysis if the effects of diffusion on our model is undertaken. We will present several classes of solutions to the diffusion equation using the various equations of state.

2 The model of a dynamic and radiating relativistic fluid star

Any isolated spherically symmetric astrophysical star is a combination of three distinct concentric zones: the innermost zone is the stellar interior where there are two component matter sources, namely null fluid matter along with radiation. The middle zone is purely a radiation zone while the outermost zone is the vacuum Schwarzschild exterior that extends roughly to a radius of 1 light year (for solar mass stars) beyond which galactic dynamics take over. In this section we briefly outline how to model all three of these zones under a combined framework using a generalised Vaidya class of metric.

2.1 Stellar interior

As described earlier, the best possible candidate for the spacetime of a stellar interior is the class of generalised Vaidya spacetimes (1), and the mass function $m(v, \mathsf{r})$ can be uniquely obtained via the Einstein field equations with the two component matter sources. Let $m(v, \mathsf{r})$ be one such solution for a given combination of fluid and radiation fields. This solution then completely describes the solution of the interior of the star, up to a boundary layer given by $\mathsf{r} = \mathsf{r}_b$. Beyond this boundary we enter a pure radiation zone.

2.2 Radiation zone

In this zone the matter field is a single component null matter field and the spacetime is well described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m_1(v)}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (2)$$

We can naturally relate the Vaidya mass function $m_1(v)$ in the radiation zone to the generalised Vaidya mass function in the stellar interior in the following way

$$m_1(v) = m(v, \mathsf{r}_b). \qquad (3)$$

This radiation zone continues until some retarded null coordinate value $v = V_0$, beyond which the spacetime is Schwarzschild (as dictated by Birkhoff's theorem).

2.3 Schwarzschild exterior

This region is well described by the exterior static subset of the completely extended Schwarzschild manifold, and the metric is given by

$$ds^2 = -\left(1 - \frac{2M}{\mathsf{r}}\right) dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4}$$

Here the Schwarzschild mass M is related to the Vaidya mass $m_1(v)$ by

$$M = m_1(V_0). \tag{5}$$

2.4 Matching conditions at the boundary layers: complete mass function

We note here that the spacetime is divided into three distinct regions for our above mentioned stellar model: the interior region, the radiation zone and the Schwarzschild exterior region. The first boundary layer between the inner and the intermediate zone, given by $\mathsf{r} = \mathsf{r}_b$, is a timelike boundary, whereas the second boundary given by $v = V_0$ is a null boundary. *The important point that all the three zones are described by the same class of metric which makes the matching conditions across these boundaries extremely transparent.* To match the first fundamental form all we need is the mass function to be continuous across these boundaries. Hence the complete C^2 mass function for an isolated stellar model can be given in the following form:

$$m(v, \mathsf{r}) = \begin{cases} m(v, \mathsf{r}) & \mathsf{r} \leq \mathsf{r}_b \ , v \leq V_0 \\ m_1(v) \equiv m(v, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v \leq V_0 \\ M \equiv m_1(V_0) \equiv m(V_0, \mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b \ , v > V_0 \end{cases} \tag{6}$$

We can easily check that this mass function is a solution to the Einstein field equations in all the three zones mentioned above, and hence it completely describes the spacetime of an isolated collapsing star. To match the second fundamental form, we need the partial derivatives of the mass functions across the boundaries be continuous. These conditions are given by

$$\frac{\partial}{\partial v} m(v, \mathsf{r}_b) = \frac{\partial}{\partial v} m_1(v), \tag{7a}$$

$$\frac{\partial}{\partial \mathsf{r}} m(v, \mathsf{r})\bigg|_{\mathsf{r}=\mathsf{r}_b} = 0, \tag{7b}$$

$$\frac{\partial}{\partial v} m_1(v)\bigg|_{v=V_0} = 0. \tag{7c}$$

where $\mathsf{r} = \mathsf{r}_b$ is the timelike boundary [from equating (1)–(2)] and $v = V_0$ is the null boundary [(from equating (2)–(4)]. These boundaries serve as the matching surfaces for the three concentric regions which can be seen in Fig. 1.

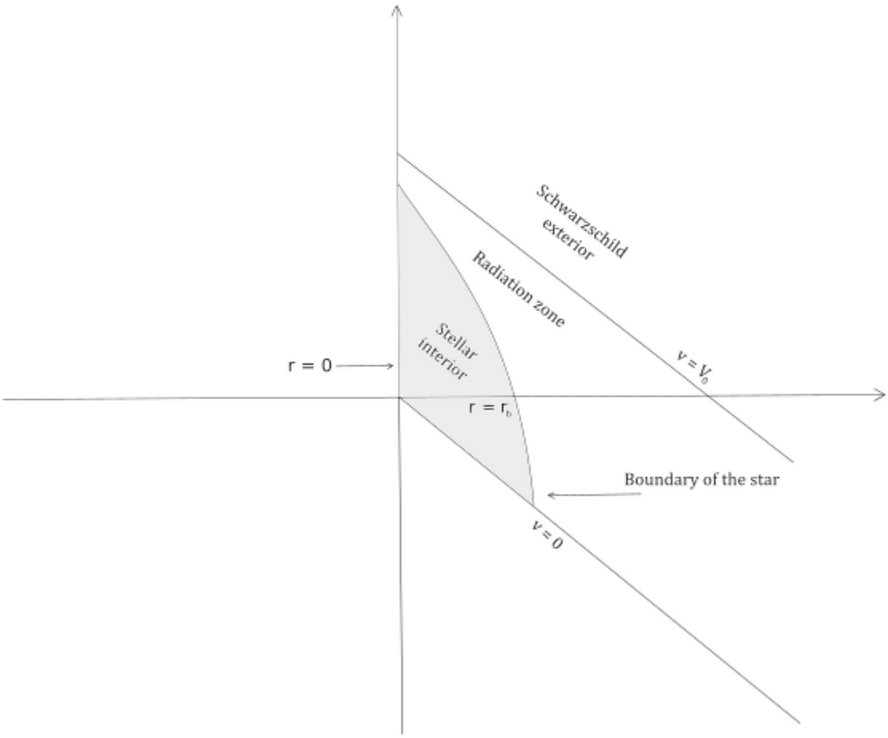

Fig. 1 Depiction of spacetime divided into the three distinct regions

It is therefore necessary to find physically relevant mass functions, with the structure of (6), to model a dynamical radiating star which is isolated. We achieve this by imposing specific equations of state.

3 Generalised Vaidya spacetime: field equations and energy conditions

The line element for all three regions belongs to the generalised Vaidya class given by (1). Note that $m(v, \mathsf{r})$ is the mass of the star and is related to the gravitational energy within a given radius r [32,33]. From the above we have the following quantities

$$R^0{}_0 = R^1{}_1 = \frac{m_{\mathsf{rr}}}{\mathsf{r}}, \tag{8a}$$

$$R^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{8b}$$

$$R^2{}_2 = R^3{}_3 = \frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{8c}$$

with the Ricci scalar

$$R = \frac{2}{\mathsf{r}^2}(\mathsf{r}m_{\mathsf{rr}} + 2m_{\mathsf{r}}),$$

where we have used the notation

$$m_v = \frac{\partial m}{\partial v}, \quad m_{\mathsf{r}} = \frac{\partial m}{\partial \mathsf{r}}.$$

The Einstein tensor components are

$$G^0{}_0 = G^1{}_1 = -\frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{9a}$$

$$G^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{9b}$$

$$G^2{}_2 = G^3{}_3 = -\frac{m_{\mathsf{rr}}}{\mathsf{r}}. \tag{9c}$$

The energy momentum tensor is defined by

$$T_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab}, \tag{10}$$

where

$$T^{(n)}_{ab} = \mu l_a l_b,$$
$$T^{(m)}_{ab} = (\tilde{\rho} + P)(l_a n_b + l_b n_a) + P g_{ab}.$$

In the above

$$l_a = \delta^0_a, \quad n_a = \frac{1}{2}\left[1 - \frac{2m(v, \mathsf{r})}{\mathsf{r}}\right]\delta^0_a + \delta^1_a,$$

with $l_c l^c = n^c_c = 0$ and $l_c n^c = -1$. The null vector l^a is a double null eigenvector of the energy-momentum tensor (10). Hence the nonzero components are given by

$$T^0{}_0 = -\tilde{\rho}, \tag{11a}$$

$$T^1{}_0 = -\mu, \tag{11b}$$

$$T^2{}_2 = T^3{}_3 = P, \tag{11c}$$

The Einstein field equations ($G^a{}_b = \kappa T^a{}_b$) become

$$\mu = -2\frac{m_v}{\kappa \mathsf{r}^2}, \tag{12a}$$

$$\tilde{\rho} = 2\frac{m_{\mathsf{r}}}{\kappa \mathsf{r}^2}, \tag{12b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\kappa \mathsf{r}}, \tag{12c}$$

which describe the gravitational behaviour of a string fluid [34,35].

The energy conditions for this kind of fluid are

1. The weak and strong energy conditions:

$$\mu \geq 0, \quad \tilde{\rho} \geq 0, \quad P \geq 0 \quad (\mu \neq 0). \tag{13}$$

2. The dominant energy condition:

$$\mu \geq 0, \quad \tilde{\rho} \geq P \geq 0 \quad (\mu \neq 0). \tag{14}$$

In the case when $m = m(v)$ the above energy conditions all reduce to $\mu \geq 0$, and if $m = m(r)$, then $\mu = 0$ and the matter field becomes a Type I fluid. For the purposes of many applications, it is a requirement that the matter distribution satisfy an equation of state

$$P = P(\tilde{\rho}), \tag{15}$$

on physical grounds. Sometimes the linear γ-law equation of state

$$P = (\gamma - 1)\tilde{\rho}, \tag{16}$$

where $0 < \gamma < 1$, is assumed in cosmology when probing the dynamics of matter on galactic and extragalactic length scales. The case $\gamma = 1$ corresponds to dust (vanishing pressure); $\gamma = 2$ gives a stiff equation of state in which the speed of sound and light speed are equal; $\gamma = 4/3$ corresponds to radiation. In the limit when $\gamma = 0$, the fluid pressure is negative, $p = -\tilde{\rho}$ (since $\tilde{\rho} > 0$). This is the characteristic property of the so-called dark energy or the existence of a possible scalar field that is responsible for the accelerated expansion of the universe. Often the particular equation of state

$$P = k\tilde{\rho}^{\gamma},$$

is assumed in relativistic astrophysics; this is called the polytropic equation of state. It is commonly used to model electron degenerate and neutron degenerate gases in white dwarfs and neutron stars, respectively.

4 Solutions with equations of state

In this section we will impose various equations of state upon the system (12).

4.1 Case I(a): linear

If we assume a linear equation of state $P = k\tilde{\rho}$ to the field Eqs. (12), we have

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}} m_{\mathsf{r}} = 0, \tag{17}$$

which is a second order linear partial differential equation. Since we are differentiating with respect to one variable, we can treat it as an ordinary differential equation, in which case it's a weaker variant of the Cauchy–Euler equation. The above equation can be solved via reduction of order and has two solutions. For the case when $k = \frac{1}{2}$, the solution is given by

$$m(v, \mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v),$$

where $c_1(v)$ and $c_2(v)$ are functions of integration. For $k \neq \frac{1}{2}$ we have the solution

$$m(v, \mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v). \tag{18}$$

Hence we have (for the latter case)

$$\mu = -\left[\frac{2\dot{c}_1}{\kappa(1-2k)\mathsf{r}^{1+2k}} + \frac{2\dot{c}_2}{\kappa\mathsf{r}^2}\right], \tag{19a}$$

$$\tilde{\rho} = \frac{2c_1}{\kappa\mathsf{r}^{2+2k}}, \tag{19b}$$

$$P = \frac{2c_1 k}{\kappa\mathsf{r}^{2+2k}}. \tag{19c}$$

We have that all the energy conditions are satisfied if we have $c_1(v) \geq 0$ and $\dot{c}_2(v) < 0$.

4.2 Case I(b): generalised linear

Imposing the condition $P = k\tilde{\rho} + k_2$ yields

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}}m_{\mathsf{r}} + \kappa k_2\mathsf{r} = 0, \tag{20}$$

which can be solved via reduction of order. Letting $y(v, \mathsf{r}) = m_{\mathsf{r}}$ yields the first order equation

$$y' + \frac{2k}{\mathsf{r}}y + \kappa k_2\mathsf{r} = 0, \tag{21}$$

which in turn has the solution

$$y(v, \mathsf{r}) = \frac{-\kappa k_2}{2k+2}\mathsf{r}^2 + c_1\mathsf{r}^{-2k}, \tag{22}$$

where $c_1 = c_1(v)$ is a function of integration. Again, two cases arise. When $k = \frac{1}{2}$ the general solution for $m(v, \mathsf{r})$ is given by

$$m(v, \mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v) - \frac{\kappa k_2}{\mathsf{r}^3},$$

where $c_2 = c_2(v)$ is a further integration function. The general solution for the mass when $k \neq \frac{1}{2}$ is

$$m(v, \mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3 + \frac{c_1\mathsf{r}^{1-2k}}{1-2k} + c_2. \tag{23}$$

Table 1 Known solutions contained within the system (24)

Solution	$m(v, \mathsf{r})$	$c_1(v)$ and $c_2(v)$	k-indices
Monopole	$\frac{a\mathsf{r}}{2}$	$c_1(v) = \frac{a}{2}, c_2(v) = 0$	$k, k_2 = 0$
Charged Vaidya	$g(v) - \frac{q(v)^2}{2\mathsf{r}}$	$c_1 = \frac{q(v)^2}{2}, c_2 = g(v)$	$k = 1, k_2 = 0$
dS/AdS	$\frac{\Lambda}{6}\mathsf{r}^3$	$c_1(v) = c_2(v) = 0$	$k = $ const., $k_2 = -\frac{\Lambda(k+1)}{\kappa}$
Husain	$g(v) - \frac{q(v)}{(2k-1)\mathsf{r}^{2k-1}}$	$c_1(v) = \frac{-q(v)}{2}, c_2(v) = g(v)$	$k, k_2 = $ const.

Thus we have (for the latter case)

$$\mu = -\left[\frac{2\dot{c}_1}{\kappa(1-2k)\mathsf{r}^{1+2k}} + \frac{2\dot{c}_2}{\kappa \mathsf{r}^2}\right], \tag{24a}$$

$$\tilde{\rho} = \frac{2c_1}{\kappa \mathsf{r}^{2+2k}} - \frac{k_2}{k+1}, \tag{24b}$$

$$P = \frac{k_2}{k+1} + \frac{2c_1 k}{\kappa \mathsf{r}^{2+2k}}, \tag{24c}$$

which contains the system (19) as well as several of the seminal other cases summarised in [9]. This list of possible solutions is presented in Table 1. Again, the energy conditions are satisfied for $c_1(v) \geq 0$ and $\dot{c}_2(v) < 0$.

4.3 Case II(a): quadratic

If we impose the quadratic equation of state $P = k\tilde{\rho}^2$ on the system (12), we have

$$m_{\mathsf{rr}} + \frac{\eta}{\mathsf{r}^3} m_{\mathsf{r}}^2 = 0, \tag{25}$$

where $\eta = 4k/\kappa$. Reducing the order of the above equation with $y(v, \mathsf{r}) = m_{\mathsf{r}}$ gives

$$y' + \frac{\eta}{\mathsf{r}^3} y^2 = 0, \tag{26}$$

which is a separable equation in y. The solution is

$$y(v, \mathsf{r}) = -\frac{2\mathsf{r}^2}{c_1 \mathsf{r}^2 + \eta}, \tag{27}$$

where $c_1 = c_1(v)$ is again, an integration function. Hence, the general solution for m is

$$m(v, \mathsf{r}) = c_2 - 2\left(\frac{\mathsf{r}}{2c_1} - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1^{3/2}}\right), \tag{28}$$

where $c_2 = c_2(v)$ is a second integration function. Hence the field equations give

$$\mu = -\frac{2}{\kappa\mathsf{r}^2}\left[\dot{c}_2 + \frac{\dot{c}_1\mathsf{r}}{c_1^2} + \frac{\dot{c}_1\mathsf{r}}{2c_1\left(1+\frac{2c_1\mathsf{r}^2}{\eta}\right)} - \frac{3\sqrt{2}\sqrt{\eta}\dot{c}_1}{4\sqrt{c_1}^5}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)\right], \tag{29a}$$

$$\tilde{\rho} = \frac{4}{\kappa(2c_1\mathsf{r}^2+\eta)}, \tag{29b}$$

$$P = \frac{4\eta}{\kappa(2c_1\mathsf{r}^2+\eta)^2}. \tag{29c}$$

For the above, the energy conditions are satisfied for the following restriction: $\eta \geq 0$ or $c_1(v) \geq 0$ and for any $c_2(v)$.

4.4 Case II(b): generalised quadratic

Imposing the condition $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$ yields

$$m_{\mathsf{rr}} + \frac{\eta}{\mathsf{r}^3}m_{\mathsf{r}}^2 + \frac{2k_2}{\mathsf{r}}m_{\mathsf{r}} + k_3\kappa\mathsf{r} = 0, \tag{30}$$

which can be solved again by reducing the order. Doing so with $y(v, \mathsf{r}) = m_{\mathsf{r}}$ yields

$$y' + \frac{\eta}{\mathsf{r}^3}y^2 + \frac{2k_2}{\mathsf{r}}y + k_3\kappa\mathsf{r} = 0. \tag{31}$$

Equation (31) is a nonlinear Riccati equation. Integration yields

$$y(v, \mathsf{r}) = -\frac{1}{\eta}\left(\mathsf{r}^2\tan\left(\sqrt{k_3\kappa\eta - k_2^2 - 2k_2 - 1}(\ln\mathsf{r} - c_1)\right) \times\sqrt{k_3\kappa\eta - k_2^2 - 2k_2 - 1}\right), \tag{32}$$

where $c_1 = c_1(v)$ is a function of integration. Thus, the solution for m can be expressed as a quadrature

$$m(v, \mathsf{r}) = -\frac{1}{\eta} \int \left(\mathsf{r}^2 \tan\left(\sqrt{\zeta}\ (\ln \mathsf{r} - c_1) \right) \sqrt{\zeta} \right) d\mathsf{r} + c_2, \tag{33}$$

where $c_2 = c_2(v)$ is a second integration function. In the above we have set $\zeta = k_3\kappa\eta - k_2^2 - 2k_2 - 1$ for convenience. So we have the result

$$\mu = \frac{2}{\kappa\eta\mathsf{r}^2} \left[\frac{\partial}{\partial v} \int \left(\mathsf{r}^2 \sqrt{\zeta} \tan\left(\sqrt{\zeta}(\ln \mathsf{r} - c_1) \right) \right) d\mathsf{r} \right] + \frac{2}{\kappa\eta\mathsf{r}^2} \dot{c}_2, \tag{34a}$$

$$\tilde{\rho} = \frac{2}{\kappa\eta\mathsf{r}^2} \left(\sqrt{\zeta}\mathsf{r}^2 \tan\left(\sqrt{\zeta}(\ln \mathsf{r} - c_1) \right) \right), \tag{34b}$$

$$P = \frac{1}{\kappa\eta} \left[2\sqrt{\zeta} \tan(\sqrt{\zeta}) \times \zeta \sec^2\left(\sqrt{\zeta}(\ln \mathsf{r} - c_1) \right) \right]. \tag{34c}$$

The energy conditions are satisfied as in the previous case for any $c_2(v)$ and the condition: $\eta \geq 0$ or $c_1(v) \geq 0$.

4.5 Case III: polytropic

If we finally impose the equation of state $P = k\tilde{\rho}^{\gamma}$, we have

$$m_{\mathsf{rr}} + k\kappa \left(\frac{2}{\kappa} \right)^{\gamma} \mathsf{r}^{1-2\gamma} m_{\mathsf{r}}^{\gamma} = 0. \tag{35}$$

Reducing the order of the above equation with $y(v, \mathsf{r}) = m_{\mathsf{r}}$ gives

$$y' + k\kappa \left(\frac{2}{\kappa} \right)^{\gamma} \mathsf{r}^{1-2\gamma} y^{\gamma} = 0, \tag{36}$$

which is a separable equation in y. Therefore the solution is

$$y(v, \mathsf{r}) = \left[(\gamma + 1)k\kappa \left(\frac{2}{\kappa} \right)^{\gamma} \frac{\mathsf{r}^{2-2\gamma}}{2 - 2\gamma} + (1 - \gamma)c_1 \right]^{\frac{1}{1-\gamma}}, \tag{37}$$

where $c_1 = c_1(v)$ is a function resulting from the integration process. So the solution for the mass m is

$$m(v, \mathsf{r}) = \int \left[(\gamma+1)k\kappa \left(\frac{2}{\kappa}\right)^{\gamma} \times \frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1 \right]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2, \tag{38}$$

where $c_2 = c_2(v)$ is a second integration function. It should be noted that this solution was first presented by [29]. The field equations yield

$$\mu = -\frac{2}{\kappa \mathsf{r}^2}\left(\frac{\partial}{\partial v}\int \left[(\gamma+1)k\kappa \left(\frac{2}{\kappa}\right)^{\gamma} \times \frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1 \right]^{\frac{1}{1-\gamma}} d\mathsf{r}\right) - \frac{2}{\kappa \mathsf{r}^2}\dot{c}_2, \tag{39a}$$

$$\tilde{\rho} = \frac{2}{\kappa \mathsf{r}^2}\left[(\gamma+1)k\kappa \left(\frac{2}{\kappa}\right)^{\gamma} \frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1 \right]^{\frac{1}{1-\gamma}}, \tag{39b}$$

$$P = \frac{1}{(1-\gamma)\mathsf{r}^{2k}}\left[(\gamma+1)k\kappa \left(\frac{2}{\kappa}\right)^{\gamma} \frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1 \right]^{\frac{\gamma}{1-\gamma}} \times (\gamma+1)k\left(\frac{2}{\kappa}\right)^{\gamma}. \tag{39c}$$

A summary of the above solutions can be found in Table 2. The weak and strong energy conditions are satisfied when $c_1(v) \geq 0$ and for any $c_2(v)$. The dominant energy condition is satisfied for $c_1(v) \geq 0$ and the further restriction: $\frac{2}{\kappa r^{2k}} \geq \frac{(\gamma+1)}{(1-\gamma)}\left(\frac{2}{\kappa}\right)^{\gamma}$.

5 Higher dimensional Vaidya spacetime

Higher dimensional Vaidya spacetimes have a variety of physical applications. For example, the thermodynamics of spacetime, entropy and the existence of horizons have been studied in detail by Debnath [36]. Also Mkenyeleye et al. [37] considered gravitational collapse in higher dimensional Vaidya spacetimes. It is also interesting to note that solutions have been found in alternate theories of gravity. For example, Dominguez and Gallo [38] found families of radiating black hole solutions for various equations of state in higher dimensional Einstein–Gauss–Bonnet gravity. Collapse and other physical features are affected by the presence of higher dimensions. The N-dimensional generalised Vaidya metric is given by

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}^{N-3}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2 d\Omega^2_{N-2}, \tag{40}$$

Table 2 Equations of state and the gravitational mass

Equation of state	$P = P(\tilde{\rho})$	$m(v, \mathsf{r})$
Linear	$P = k\tilde{\rho}$	$m(v, \mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v), \quad (k = \frac{1}{2})$
		$m(v, \mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v), \quad (k \neq \frac{1}{2})$
Generalised linear	$P = k\tilde{\rho} + k_2$	$m(v, \mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3 + \frac{c_1(v)\mathsf{r}^{1-2k}}{1-2k} + c_2(v)$
Quadratic	$P = k\tilde{\rho}^2$	$m(v, \mathsf{r}) = c_2(v) - 2\left(\frac{\mathsf{r}}{2c_1(v)} - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1(v)}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1(v)^{3/2}}\right)$
Generalised quadratic	$P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$	$m(v, \mathsf{r}) = c_2(v) - \frac{1}{\eta}\int\left(\mathsf{r}^2\tan\left(\sqrt{\zeta}\,(\ln\mathsf{r} - c_1(v))\right)\sqrt{\zeta}\right)d\mathsf{r}$
Polytropic	$P = k\tilde{\rho}^{\gamma}$	$m(v, \mathsf{r}) = \int[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^{\gamma} \times \frac{\mathsf{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1(v)]^{\frac{1}{1-\gamma}}\,d\mathsf{r} + c_2(v)$

where

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta^j)\right](d\theta^i)^2.$$

The nonvanishing Ricci tensor components are given by

$$R^0{}_0 = R^1{}_1 = \frac{m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}} - \frac{(N-4)m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, \tag{41a}$$

$$R^1{}_0 = \frac{(N-2)m_v}{\mathsf{r}^{(N-2)}}, \tag{41b}$$

$$R^2{}_2 = R^3{}_3 = \cdots = R^{\theta(N-2)}{}_{\theta(N-2)} = \frac{2m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, \tag{41c}$$

with the Ricci scalar

$$R = \frac{2m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}} + \frac{4m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}. \tag{42}$$

The nonvanishing components of the Einstein tensor are

$$G^0{}_0 = G^1{}_1 = -\frac{(N-2)m_{\mathsf{r}}}{\mathsf{r}^{(N-2)}}, \tag{43a}$$

$$G^1{}_0 = \frac{(N-2)m_v}{\mathsf{r}^{(N-2)}}, \tag{43b}$$

$$G^2{}_2 = G^3{}_3 = \cdots = G^{\theta(N-2)}{}_{\theta(N-2)} = -\frac{m_{\mathsf{rr}}}{\mathsf{r}^{(N-3)}}. \tag{43c}$$

Table 3 Equations of state and the higher dimensional gravitational mass

Equation of state	$P = P(\tilde{\rho})$	$m(v, \mathsf{r})$
Linear	$P = k\tilde{\rho}$	$m(v, \mathsf{r}) = c_1(v)\ln(\mathsf{r}) + c_2(v), \quad (k = \frac{1}{N-2})$ $m(v, \mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v), \quad (k \neq \frac{1}{N-2})$
Generalised linear	$P = k\tilde{\rho} + k_2$	$m(v, \mathsf{r}) = -\frac{\kappa k_2}{(N-2)k+N-2}\frac{\mathsf{r}^{N-1}}{N-1} + c_1(v)\frac{\mathsf{r}^{1-(N-2)k}}{1-(N-2)k} + c_2(v)$
Quadratic	$P = k\tilde{\rho}^2$	$m(v, \mathsf{r}) = (2-N)\int \frac{\mathsf{r}^{N-2}}{c_1(v)(N-2)\mathsf{r}^{N-2}+\eta} d\mathsf{r} + c_2(v)$
Generalised quadratic	$P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$	$m(v, \mathsf{r}) = -\frac{1}{2\eta}\int \Big[\Big(\mathsf{r}^{N-2}\tan(\sqrt{\varsigma}(\ln \mathsf{r} - c_1(v))\Big)$ $\times(\sqrt{\varsigma} + N - 2 + \xi)\Big] d\mathsf{r} + c_2(v)$
Polytropic	$P = k\tilde{\rho}^{\gamma}$	$m(v, \mathsf{r}) = \int \Big[\kappa k(\gamma+1)\left(\frac{N-2}{\kappa}\right)^{\gamma} \times \frac{\mathsf{r}^{N-2-\gamma(N-2)}}{N-2-\gamma(N-2)}$ $+(1-\gamma)c_1(v)\Big]^{\frac{1}{1-\gamma}} d\mathsf{r} + c_2(v)$

The Einstein field equations are thus

$$\mu = -\frac{(N-2)m_v}{\kappa \mathsf{r}^{N-2}}, \tag{44a}$$

$$\tilde{\rho} = \frac{(N-2)m_{\mathsf{r}}}{\kappa \mathsf{r}^{N-2}}, \tag{44b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\kappa \mathsf{r}^{N-3}}. \tag{44c}$$

As in Sect. 4 we can find solutions to the field equations with various equations of state for the higher dimensional Vaidya spacetime (40). The results are presented in Table 3 for particular equations of state.

6 Diffusion

The notion of diffusion is an important one in regards to the understanding of many physical systems. The ideas of diffusion have been applied to fields as diverse as the stock exchange, kinetic theory and physiology. Vilenken [39] characterised string evolutions as the formation of Brownian trajectories in an attempt to introduce diffusion into the description of cosmic strings. Calogero [40] presented a new model to describe the dynamics of particles undergoing diffusion in general relativity. It was shown that in the flat Robertson–Walker spacetime, either unlimited expansion or the formation of a singularity may occur, depending on the initial value of the cosmological scalar field. If we assume that string diffusion is likened to point particle diffusion then we have

$$\frac{\partial}{\partial v} n = \mathcal{D}\nabla^2 n, \tag{45}$$

where $\nabla^2 = r^{-2}\left(\frac{\partial}{\partial r}\right) r^2 (\frac{\partial}{\partial r})$ and $\mathcal{D}$ is the positive coefficient of self-diffusion, which we treat as a constant. In classical transport theory the diffusion equation is derived beginning with Fick's law

$$\mathbf{J}_{(n)} = -\mathcal{D}\boldsymbol{\nabla} n, \tag{46}$$

where $\boldsymbol{\nabla}$ is a purely spatial gradient. The 4-current conservation $J^a_{(n);a} = 0$, where

$$\begin{aligned} J^a_{(n);a} &= (n, \mathbf{J}_{(n)}) \\ &= n\frac{\partial}{\partial u} - \mathcal{D}\left(\frac{\partial n}{\partial \mathsf{r}}\right)\left(\frac{\partial}{\partial \mathsf{r}}\right), \end{aligned} \tag{47}$$

then yields the diffusion Eq. (45). Rewriting the field Eqs. (12a) and (12b) as $m_v = -\kappa\mu\mathsf{r}^2$ and $m_{\mathsf{r}} = \kappa\tilde{\rho}\mathsf{r}^2$, we can express the integrability condition for m as

$$\frac{\partial\tilde{\rho}}{\partial v} + \frac{1}{\mathsf{r}^2}\frac{\partial}{\partial \mathsf{r}}(\mathsf{r}^2\mu) = 0. \tag{48}$$

If we compare the diffusion Eq. (45) (n replaced with ρ) with $\tilde{\rho}_v$ in Eq. (48) above, we get

$$\frac{\partial m}{\partial v} = \mathcal{D}\mathsf{r}^2\frac{\partial\tilde{\rho}}{\partial \mathsf{r}}. \tag{49}$$

Solving the above Eq. (49) for the mass function $m(v, \mathsf{r})$ will provide solutions for the Einstein equations. Recall that the equations of state presented earlier were of the form $F_1(m', m'') = 0$. Using the field Eq. (12b) and substituting into (49) we get the following

$$\frac{\partial m}{\partial v} = \frac{2\mathcal{D}}{\kappa}\left(\frac{\partial^2 m}{\partial \mathsf{r}^2} - \frac{4}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}}\right), \tag{50}$$

which is of the functional form $F_2(\dot{m}, m', m'') = 0$. In order to solve (50) entirely we require a functional form $F_3(\dot{m}, m') = 0$. This entails isolating $\frac{\partial^2 m}{\partial \mathsf{r}^2}$ in each equation of state and substituting into (50). We will consider some cases below.

6.1 Linear

If we begin with the linear equation of state $P = k\rho$ we have from before

$$m_{\mathsf{rr}} + \frac{2k}{\mathsf{r}} m_{\mathsf{r}} = 0,$$

which we can substitute into (50) to finally get

$$\frac{\partial m}{\partial v} - \frac{\alpha}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} = 0, \tag{51}$$

where $\alpha = -\frac{2\mathcal{D}}{\kappa}(2k+4)$ for convenience. Equation (51) can be solved using the method of characteristics and the solution is given by

$$m(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right), \tag{52}$$

which is an infinite family of solutions. To check for consistency, we simply have to substitute (52) into (17). In doing so we get

$$\mathsf{r}^2\mathcal{F}'' + (1+2k)\mathcal{F}' = 0, \tag{53}$$

which is a consistency condition on $\mathcal{F}$. It turns out that a solution is only possible for the case when $k = -\frac{1}{2}$. It is given by

$$\mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) = l_1\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) + l_2, \tag{54}$$

where l_1 and l_2 are constants.

6.2 Generalised linear

For the generalised linear equation of state the resulting partial differential equation becomes

$$\frac{\partial m}{\partial v} - \frac{\alpha}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} + \kappa k_2 \mathsf{r} = 0, \tag{55}$$

and can be solved in the same way as above giving the solution

$$m(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}^2 + \alpha v\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}^3. \tag{56}$$

The above solution is consistent if and only if

$$\mathsf{r}^2\mathcal{F}'' + (1+2k)\mathcal{F}' - \left(\frac{2k_2\kappa}{\alpha} + \frac{2kk_2\kappa}{\alpha} - k_2\kappa\right)\mathsf{r} = 0. \tag{57}$$

Then we must have

$$\mathcal{F}'' = (1+2k)\mathcal{F}' = \left(\frac{2k_2\kappa}{\alpha} + \frac{2kk_2\kappa}{\alpha} - k_2\kappa\right) = 0. \tag{58}$$

This implies that $\mathcal{F}$ has the same form as (54) and $k = -\frac{1}{2}$. We also have

$$k_2\kappa\left(1 - \frac{1}{\alpha}\right) = 0. \tag{59}$$

Since $k_2 \neq 0$, we have that $\left(1-\frac{1}{\alpha}\right)=0$ which implies $\alpha = 1$ and so $\kappa = -6\mathcal{D}$. This is also a generalisation of the first result (54) where $k = -\frac{1}{2}$ with the added restriction that $\kappa = -6\mathcal{D}$.

6.3 Generalised quadratic

Isolating m_{rr} in the Eq. (30) and substituting into (50) the resulting partial differential equation is given by

$$\frac{\partial m}{\partial v} - \Theta\eta\frac{1}{\mathsf{r}^3}\left(\frac{\partial m}{\partial \mathsf{r}}\right)^2 - \Theta\left(\frac{2k_2+4}{\mathsf{r}}\right)\frac{\partial m}{\partial \mathsf{r}} - \Theta k_3\kappa\mathsf{r} = 0, \tag{60}$$

where $\Theta = -\frac{2\mathcal{D}}{\kappa}$ and $\eta = \frac{4k}{\kappa}$. This equation above cannot be solved via the method of characteristics and so another approach is needed. If we assume a separable solution for the mass function m of the form

$$m(v,\mathsf{r}) = a(v) + b(\mathsf{r}),$$

then we can express (60) as two ordinary differential equations

$$\frac{da}{dv} = c, \tag{61a}$$

$$\Theta\eta\frac{1}{\mathsf{r}^3}\left(\frac{db}{d\mathsf{r}}\right)^2 - \Theta\left(\frac{2k_2+4}{\mathsf{r}}\right)\frac{db}{d\mathsf{r}} - \Theta k_3\kappa\mathsf{r} = c, \tag{61b}$$

where c is a constant. Both of these equations can be analysed independently and used to yield a solution for the master Eq. (60). Solving Eqs. (61a) and (61b) yields the final expression for the mass function as

$$\begin{aligned} m(v,\mathsf{r}) = cv + \varepsilon - \frac{(2k_2+4)}{6\eta}\mathsf{r}^3 \pm \Bigg[&\frac{1}{6\alpha\beta\eta}(\alpha\mathsf{r}^2+\beta\mathsf{r})^{\frac{3}{2}} \\ &-\frac{\beta\sqrt{\alpha\mathsf{r}}}{8\alpha^{\frac{5}{2}}\eta\sqrt{\beta}}\left(1+\frac{\alpha\mathsf{r}}{\beta}\right)^{\frac{3}{2}} + \frac{\beta}{16\alpha^{\frac{5}{2}}\eta}\sqrt{1+\frac{\alpha\mathsf{r}}{\beta}} \\ &-\frac{\beta}{2\alpha^{\frac{5}{2}}\eta}\ln\left(\sqrt{1+\frac{\alpha\mathsf{r}}{\beta}}+\frac{\sqrt{\alpha\mathsf{r}}}{\sqrt{\beta}}\right) + \frac{\zeta}{2\beta\eta}\Bigg], \end{aligned} \tag{62}$$

where $\alpha = \Theta^2(2k_2+4)^2 - 4\Theta^4 k_3\kappa$, $\beta = 4c$ and, as before $\eta = \frac{4k}{\kappa}$. In the above ε and ζ are constants of integration. The mass functions (62) are other solutions to the diffusion Eq. (50) which we believe are new. The functional form (62) has to be consistent with the equation of state (30).

6.4 Polytropic

Special mention should be made about this particular case. The partial differential equation resulting from substitution of the expression (35) for the polytrope is

$$\frac{\partial m}{\partial v} - \beta\kappa k\left(\frac{2}{\kappa}\right)\mathsf{r}^{2-2\gamma}\left(\frac{\partial m}{\partial \mathsf{r}}\right)^{\gamma} - \beta\frac{4}{\mathsf{r}}\frac{\partial m}{\partial \mathsf{r}} = 0, \tag{63}$$

which is a first order, degree γ nonlinear equation. The above equation can only be solved for specific values of the constant γ but not in general. Specifically, we have shown that it only admits general closed form analytical solutions only for $0 < \gamma \leq 2$. The case $\gamma > 3$ is highly nonlinear and not easy to analyse.

7 Discussion

In this work we considered a spherically symmetric radiating star. We noted that any astrophysical star is a combination of three distinct concentric zones: the innermost two-component matter zone, the middle radiation zone and the outermost zone which is the vacuum Schwarzschild exterior. A large family of solutions to the field equations were presented for various thermodynamically realistic equations of state. We showed that it was possible to obtain solutions via a direct integration of simple second order differential equations. Note that many of our solutions cannot be found using the Wang and Wu [9] approach; they assumed a series form of the mass function which is restrictive. Several other mass functions have been shown to exist in four and higher dimensions which are physically reasonable. It was also possible to obtain several diffusive solutions for the mass function via a substitution of the above mentioned second order equations into the diffusion equation. We can easily show that a dynamical radiating star is possible by matching the mass function (6) at the two boundaries. We illustrate this with the generalised linear equation of state

$$P = k\tilde{\rho} + k_2. \tag{64}$$

At the first interface $\mathsf{r} = \mathsf{r}_b$, between the two-component region and the null Vaidya zone, the mass function is

$$m_1(v) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}_b^3 + \frac{c_1(v)\mathsf{r}_b^{1-2k}}{1-2k} + c_2(v). \tag{65}$$

At the second interface, between the Vaidya zone and the vacuum exterior the mass function is

$$M = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}_b^3 + \frac{c_1(V_0)\mathsf{r}_b^{1-2k}}{1-2k} + c_2(V_0). \tag{66}$$

Clearly the forms (65) and (66) are always possible since $c_1(v)$ and $c_2(v)$ are arbitrary functions. A comparison with earlier well known results was undertaken and we showed that our solutions generalise all of the earlier ones, including those of Husain

[29]. We then generalised our results to higher dimensional spacetimes. Additionally, diffusive solutions are also possible at each interface for the same reasons. These (for the generalised linear case) are given by

$$m_1(v, \mathsf{r}) = \mathcal{F}\left(\frac{1}{2}\mathsf{r}_b^2 + \alpha v\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}_b^3, \tag{67a}$$

$$M = \mathcal{F}\left(\frac{1}{2}\mathsf{r}_b^2 + \alpha V_0\right) - \frac{\kappa k_2}{3\alpha}\mathsf{r}_b^3, \tag{67b}$$

at the first and second interfaces respectively.

Another important observation that transparently comes out of our analysis here is the nonlinear nature of gravity. Even though the energy momentum tensor can be written as a combination of radiation and matter parts, these quantities intertwine in the metric in such a way as to give physically interesting solutions that can model a dynamic star. If we switch off the radiation part completely, then the field equations force the remaining matter to obey an equation of state $\tilde{\rho} + p_r = 0$ (p_r is the radial pressure), which is that of an anisotropic de Sitter like space, and hence not appropriate for stellar modeling.

The work in this paper can further be enhanced by considering the notion of gravitational collapse; whether or not there are special classes of Vaidya mass functions for which a collapse comes to an end as a naked singularity or not. This will be a future endeavour.

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References

1. Vaidya, P.C.: Proc. Indian Acad. Sci. A **33**, 264 (1951)
2. Petrov, A.Z.: Gen. Relativ. Gravit. **32**, 8 (2002)
3. de Oliveira, A.K.G., Santos, N.O., Kolassis, C.A.: Mon. Not. R. Astron. Soc. **216**, 1001 (1985)
4. de Oliveira, A.K.G., de Pacheco, J.A.F., Santos, N.O.: Mon. Not. R. Astron. Soc. **220**, 405 (1986)
5. de Oliveira, A.K.G., Kolassis, C.A., Santos, N.O.: Mon. Not. R. Astron. Soc. **231**, 1011 (1988)
6. Kramer, D.: J. Math. Phys. **33**, 1458 (1992)
7. Govender, M., Maharaj, S.D., Maartens, R.: Class. Quantum Gravity **15**, 323 (1998)
8. Santos, N.O.: Mon. Not. R. Astron. Soc. **216**, 403 (1985)
9. Wang, A., Wu, Y.: Gen. Relativ. Gravit. **31**, 107 (1999)
10. Maharaj, S.D., Govender, G., Govender, M.: Gen. Relativ. Gravit. **44**, 1089 (2012)
11. de Oliveira, A.K.G., Santos, N.O.: Astrophys. J. **312**, 640 (1987)
12. Maharaj, S.D., Govender, G.: Pramana J. Phys. **54**, 715 (2000)
13. Pinheiro, G., Chan, R.: Gen. Relativ. Gravit. **45**, 243 (2013)
14. Schäfer, D., Goenner, H.F.: Gen. Relativ. Gravit. **32**, 2119 (2000)
15. Misthry, S.S., Maharaj, S.D., Leach, P.G.L.: Math. Methods Appl. Sci. **31**, 363 (2008)
16. Abebe, G.Z., Maharaj, S.D., Govinder, K.S.: Gen. Relativ. Gravit. **46**, 1650 (2014)
17. Abebe, G.Z., Maharaj, S.D., Govinder, K.S.: Gen. Relativ. Gravit. **46**, 1733 (2014)
18. Govender, G., Brassel, B.P., Maharaj, S.D.: Eur. Phys. J. C **75**, 324 (2015)
19. Maharaj, S.D., Govender, M.: Int. J. Mod. Phys. D **14**, 667 (2005)
20. Martinez, J.: Phys. Rev. D **53**, 6921 (1996)

21. Herrera, L., Santos, N.O.: Mon. Not. R. Astron. Soc. **287**, 161 (1997)
22. Govender, M., Maartens, R., Maharaj, S.D.: Mon. Not. R. Astron. Soc. **310**, 557 (1999)
23. Naidu, N.F., Govender, M., Govinder, K.S.: Int. J. Mod. Phys. D **15**, 1053 (2006)
24. Naidu, N.F., Govender, M.: J. Astrophys. Astron. **28**, 167 (2008)
25. Maharaj, S.D., Govender, G., Govender, M.: Pramana J. Phys. **77**, 469 (2011)
26. Mkenyeleye, M.D., Goswami, R., Maharaj, S.D.: Phys. Rev. D **90**, 064034 (2014)
27. Wagh, S.M., Govender, M., Govinder, K.S., Maharaj, S.D., Muktibodh, P.S., Moodley, M.: Class. Quantum Gravity **18**, 2147 (2001)
28. Goswami, R., Joshi, P.S.: Class. Quantum Gravity **21**, 3645 (2004)
29. Husain, V.: Phys. Rev. D **53**, R1759 (1996)
30. Dawood, A.K., Ghosh, S.G.: Phys. Rev. D **70**, 104010 (2004)
31. Ghosh, S.G., Dawood, A.K.: Gen. Relativ. Gravit. **40**, 9 (2008)
32. Lake, K., Zannias, T.: Phys. Rev. D **43**, 1798 (1991)
33. Poisson, E., Israel, W.: Phys. Rev. D **41**, 1796 (1990)
34. Glass, E.N., Krisch, J.P.: Phys. Rev. D **57**, R5945 (1998)
35. Glass, E.N., Krisch, J.P.: Class. Quantum Gravity **16**, 1175 (1999)
36. Debnath, U.: Int. J. Theor. Phys. **53**, 2108 (2014)
37. Mkenyeleye, M.D., Goswami, R., Maharaj, S.D.: Phys. Rev. D **92**, 024041 (2015)
38. Dominguez, A.E., Gallo, E.: Phys. Rev. D **73**, 064018 (2006)
39. Vilenken, A.: Phys. Rev. D **24**, 2082 (1981)
40. Calogero, S.: J. Cosmol. Astropart. Phys. **2011**, 016 (2011)