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A relativistic heat conducting model

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Abstract. The interior dynamics of a relativistic fluid in a shear-free spherically symmetric spacetime are investigated. The isotropic matter distribution is an imperfect fluid with a nonvanishing heat flux which is in the radial direction. The pressure isotropy condition is a second-order nonlinear ordinary differential equation with variable coefficients in the gravitational potentials. We impose a particular on these potentials and a new class of solutions are obtained, containing those of Bergmann and Modak. A physical analysis is then performed where the matter variables are graphically plotted and the energy conditions are shown to be satisfied. Causality is also shown not to be violated. An analysis of the temperature profiles indicates that closed form expressions can be generated for both the noncausal and causal cases.

1 Introduction

Shear-free spacetimes are extensively used to model the interior of relativistic stars which, in the form of radial heat flow, dissipate null radiation. The heat flows outward from the much hotter centre toward the surface of the star. Investigations in this direction include the models by Santos [1], Glass [2], Deng and Mannheim [3,4], Stephani *et al.* [5] and Ivanov [6]. Wagh *et al.* [7] generated models for a spherically symmetric shear-free spacetime with nonvanishing heat flux, by imposing a barotropic equation of state. Herrera *et al.* [8] discovered analytical solutions to the Einstein field equations for fluid spheres undergoing collapse in the diffusion approximation. Abebe *et al.* [9] used the approach of Lie symmetries to generate shear-free radiating stars containing metrics that satisfy a linear barotropic equation of state. Maharaj *et al.* [10] investigated the gravitational collapse of a star which eventually evolves into a final static configuration described by the interior Schwarzschild solution. They demonstrated the remarkable application of causal thermodynamics in modelling the thermal evolution of the compact object. The study of relativistic thermodynamics in radiating stars has been extensively pursued in [8,11–13]. In many of these prior models, the shear-free dissipative gravitational collapse of stars was described in the form of radial heat flow in the so-called free streaming approximation. It was shown that the relaxational effects are important during late stages of collapse and that these effects lead to much higher temperatures within the interior of the star. Closer to the surface the temperature can be lower. The earlier Eckart [14] framework for heat transport has deficiencies. This is due mainly to the noncausal nature of the theory. Eckart first extended irreversible thermodynamics from Newtonian to relativistic fluids, however its main deficiency is the fact that dissipative perturbations propagated at infinite speeds which was unreasonable in a relativistic theory. This has led to the use of causal heat transport equations that have the Maxwell-Cattaneo form. As mentioned by [15], extended irreversible thermodynamics arises from the fact that an extended set including the dissipative variables was required to study nonequilibrium states. This notion leads to stable and causal behaviour under a wide variety of conditions. Israel and Stewart [16,17] studied the relativistic version of this theory, and this is called causal/second-order thermodynamics (due to the dissipative variables being of second order in entropy) and transient thermodynamics (transient phenomena were included in the theory outside the quasi-stationary system of the classical theory). The main difference between the Eckart and Israel-Stewart equations is that the former are mere algebraic equations while the latter are differential evolution equations which are much more difficult to solve in general.

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This paper is organised as follows: in sect. 2 we present the model for a radiating shear-free interior matter distribution with heat flux. We generate, in sect. 3, a new solution for the pressure isotropy condition by assuming a power law form for one of the gravitational potentials. We show that our solution contains some of the classic solutions found previously. In sect. 4 we show an analysis of the physics of our solution, and generate plots for the energy density, pressure and heat flux. We also show that the energy conditions as well as causality are not violated. Section 5 deals with the relativstic thermodynamics of our model. A brief description of the Maxwell-Cattaneo heat transport equation is presented and we then generate the temperature profiles with potentials from the shear-free metric in general. The causal and noncausal temperature profiles are obtained for this new solution.

2 The model

The metric for spherically symmetric models, in the absence of shear, has the form

$$\mathrm{d}s^2 = -A^2\mathrm{d}t^2 + B^2[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]. \tag{1}$$

We have used coordinates $(x^a) = (t, r, \theta, \phi)$ which are isotropic and comoving. The potentials A and B are functions of both the coordinate t and the coordinate r. The matter distribution for a heat conducting relativistic fluid has the energy momentum tensor

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab} + q^a u^b + q^b u^a, \tag{2}$$

where ρ is the energy density, p is the isotropic pressure and $\mathbf{u}$ is the heat flux vector $(q^a u_a) = 0$. The comoving fluid four-velocity $\mathbf{u}$ is unit and timelike $(u^a u_a = -1)$. The Einstein field equations $G_{ab} = T_{ab}$ then take the form

$$\rho = \frac{3\dot{B}^2}{A^2B^2} - \frac{1}{B^2}\left(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\right), \tag{3a}$$

$$p = \frac{1}{A^2}\left(\frac{-2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB}\right) + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2A'B'}{AB} + \frac{2A'}{rA} + \frac{2B'}{rB}\right), \tag{3b}$$

$$p = -\frac{2\ddot{B}}{BA^2} + \frac{2\dot{A}\dot{B}}{BA^3} - \frac{\dot{B}^2}{A^2B^2} + \frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{A''}{AB^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3}, \tag{3c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right). \tag{3d}$$

This is a system of coupled partial differential equations which are highly nonlinear. It is necessary to integrate the system (3) to describe the dynamics of the matter field with heat flux.

The fluid pressure is isotropic, and eqs. (3b) and (3c) generate a consistency condition. We obtain the differential equation

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right) \tag{4}$$

governing the gravitational behaviour of the radiating spacetime. If we define the new variable

$$x = r^2,$$

the pressure isotropy condition (4) then has the equivalent form

$$\left(\frac{1}{B}\right)A_{xx} + 2A_x\left(\frac{1}{B}\right)_x - A\left(\frac{1}{B}\right)_{xx} = 0, \tag{5}$$

in terms of the new variable x. Equation (5) governs the dynamics of a shear-free spherically symmetric model with heat flow. Several solutions to (5) were presented by Brassel *et al.* [18].

3 Exact solutions

We observe that (5) admits the simple solution

$$A = 1 + M(t)x, \qquad B = R(t). \tag{6}$$

This exact model was first found by Modak [19]. Clearly (5) admits other solutions and our objective is to find another class that has a more general form than that of Modak. A generalisation of the Modak solution is a power law function depending on the variable x which is the form that we assume below. It is then possible to generate an Cauchy-Euler equation for a suitable choice of $\frac{1}{B}$. We assume the following functional form:

$$\frac{1}{B} = \alpha(a+bx)^k, \tag{7}$$

where a, $b \neq 0$ and α are functions of time and $k \neq 0$ is a real parameter. Then the condition (5) becomes

$$(a+bx)^2 A_{xx} + 2bk(a+bx)A_x - b^2k(k-1)A = 0. \tag{8}$$

We can simplify (8), by introducing the new dependent variable

$$z = a + bx.$$

Then (8) reduces to the Cauchy-Euler differential equation

$$z^2 b^2 \tilde{A}_{zz} + 2b^2 kz\tilde{A}_z - b^2k(k-1)\tilde{A} = 0, \tag{9}$$

where $\tilde{A} = \tilde{A}(z,t)$. The characteristic equation corresponding to (9) is

$$m^2 + (2k-1)m - (k^2 - k) = 0.$$

The roots of the characteristic equation are

$$m_1 = \frac{(1-2k) + \sqrt{8k^2 - 8k + 1}}{2}, \qquad m_2 = \frac{(1-2k) - \sqrt{8k^2 - 8k + 1}}{2}.$$

Three cases arise depending on the value of $8k^2 - 8k + 1$ which could be positive, negative or zero.

i) *Repeated roots.* If $k = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$ or $k = \frac{1}{2}(1 - \frac{1}{\sqrt{2}})$ then the roots are repeated, and $m_1 = m_2 = \frac{1}{2} - k$. Then the solution of (9) is given by

$$\tilde{A} = [c + d\ln z]\, z^{(1-2k)/2}.$$

In terms of the variable x we have

$$A(x,t) = [c + d\ln(a+bx)]\,(a+bx)^{(1-2k)/2}, \tag{10}$$

where $c(t)$ and $d(t)$ are functions of integration.

ii) *Real distinct roots.* If $\frac{1}{2}(1 - \frac{1}{\sqrt{2}}) < k < \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$ then the roots m_1 and m_2 are real and distinct and the solution of (9) is

$$\tilde{A} = cz^{[-(2k-1)+\sqrt{8k^2-8k+1}]/2} + dz^{[-(2k-1)-\sqrt{8k^2-8k+1}]/2}, \tag{11}$$

where $c(t)$ and $d(t)$ result from integration. The closed form solution in terms of x is given by

$$A(x,t) = c(a+bx)^{[(1-2k)+\sqrt{8k^2-8k+1}]/2} + d(a+bx)^{[(1-2k)-\sqrt{8k^2-8k+1}]/2}, \tag{12}$$

for this case.

iii) *Complex roots.* If $\frac{1}{2}(1 + \frac{1}{\sqrt{2}}) < k < \frac{1}{2}(1 - \frac{1}{\sqrt{2}})$ then the roots m_1 and m_2 are complex and the solution of (9) is

$$\tilde{A} = e^{(1-2k)(z-a)/2b}\left[c\cos\sqrt{8k^2-8k+1}\left(\frac{z-a}{b}\right) + d\sin\sqrt{8k^2-8k+1}\left(\frac{z-a}{b}\right)\right], \tag{13}$$

where $c(t)$ and $d(t)$ are functions of integration. Then the closed form solution in terms of the original variable x is given by

$$A(x,t) = e^{(1-2k)x/2}\left[c\cos\sqrt{8k^2-8k+1}x + d\sin\sqrt{8k^2-8k+1}x\right], \tag{14}$$

for complex roots.

Hence we have generated a new class of solutions in terms of elementary functions, to the condition of pressure isotropy (5). We can present the solution in the compact form

$$A = \begin{cases} [c + d\ln(a+bx)]\,(a+bx)^{(1-2k)/2}, & k = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \text{ or } \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \\ c(a+bx)^{[(1-2k)+\sqrt{8k^2-8k+1}]/2} + d(a+bx)^{[(1-2k)-\sqrt{8k^2-8k+1}]/2}, & \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \\ e^{(1-2k)x/2}\left[c\cos\sqrt{8k^2-8k+1}x + d\sin\sqrt{8k^2-8k+1}x\right], & \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) < k < \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right) \end{cases}, \quad (15)$$

$$\frac{1}{B} = (a+bx)^k,$$

for the gravitational potentials A and B.

We can regain special cases found previously from our new general class of models in (15). We show this with two examples. Firstly, in the special case when $k = 1$, the roots are real and distinct. Then the above solution yields the particular case

$$A = \frac{(ca+d)+cbx}{a+bx}, \qquad \frac{1}{B} = a+bx. \quad (16)$$

If we make the identification $ca + d = 1$, $cb = C_1$, $b = C_2$ and $a = C_3$ then we observe that (16) is equivalent to the conformally flat solution. Hence we have found a new class of exact radiating models for shear-free fluids in terms of elementary functions which generalise the conformally flat case. These solutions will help in the construction of models where tidal effects are important, *e.g.* in galaxy formation. Shear-free conformally flat fluids which are heat conducting have been studied by Herrera *et al.* [12] and Msomi *et al.* [20]. Secondly, in the geodesic case when $A = 1$ with $k = 1$, we have

$$A = 1, \qquad B = \frac{\alpha(t)}{a(t)+b(t)x}. \quad (17)$$

By then setting $\alpha(t) = R(t)$ and $a(t) = M(t)$ we regain the solution of Bergmann [21].

4 Physical analysis

In this section we test the physics of our radiating model. We will utilise the solution for repeated roots in (10). If we make the following choices: $a(t) = t^2$, $b(t) = e^t$, $c(t) = -e^{t^2}$ and $d(t) = 1$, the solution then becomes

$$A = \left[-e^{t^2} + \ln(t^2 + e^t r^2)\right](t^2 + e^t r^2)^{\frac{1}{2}(1-2k)}, \quad (18a)$$

$$B = (t^2 + e^t r^2)^{-k}, \quad (18b)$$

where $k = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$. Finding analytical expressions for the matter variables with these choices is still a difficult endeavour; however graphical plots can generated. Figures 1, 2 and 3 depict the radial profiles of the energy density, pressure and heat flux. It is evident that the general behaviour is acceptable for a cosmological model in the sense that the profiles are positive and regular in the region plotted. In figs. 4 and 5 we notice that the weak and strong energy conditions $W = \rho - p + \Delta$ and $S = 2p + \Delta$ with $\Delta = \sqrt{(p+q)^2 - 4q^2}$ are positive: hence they are satisfied. The dominant energy condition $D = \rho - 3p + \Delta$ is plotted in fig. 6. We observe that it is positive up to a certain radial coordinate which implies that it is satisfied within a certain region of the fluid. Figure 7 depicts the sound speed $c_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}$ through the fluid medium. It is evident that the curve is positive and finite throughout the interior and that causality is not violated. Finally in fig. 8 the radial profile of the adiabatic index

$$\Gamma = \left(\frac{\rho+p}{p}\right)\frac{\mathrm{d}p}{\mathrm{d}\rho},$$

is depicted. The effective adiabatic index measures the ability of the fluid matter to resist compression under gravity, and is a measure of the dynamical stability of this matter at a given instant of time. We see that $\Gamma > \frac{4}{3}$ and that the curve is positive throughout the interior of the fluid distribution.

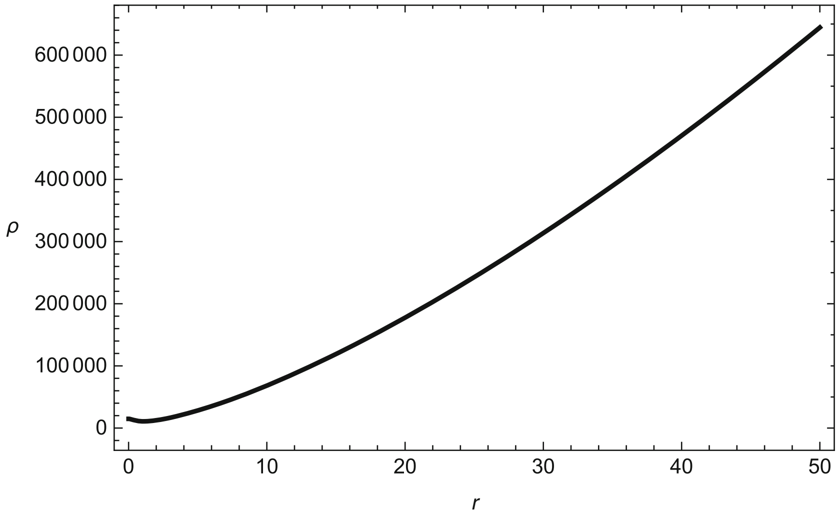

Fig. 1. Radial profile of the energy density ρ.

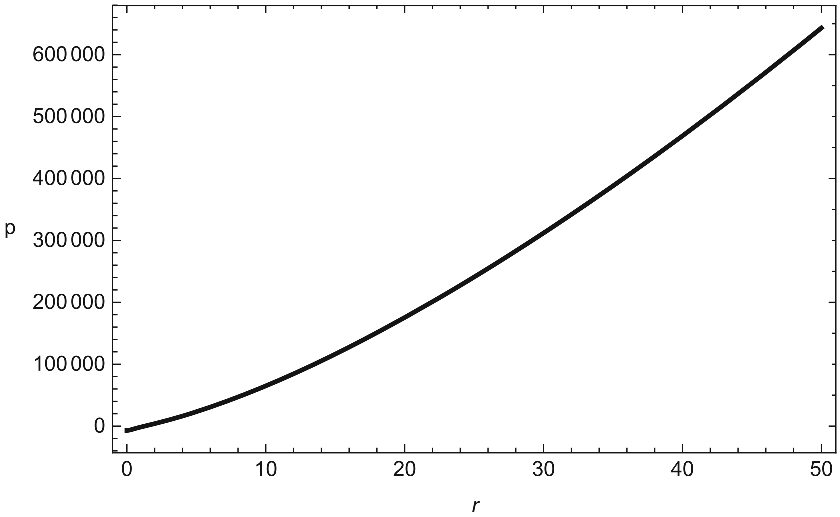

Fig. 2. Radial profile of the pressure p.

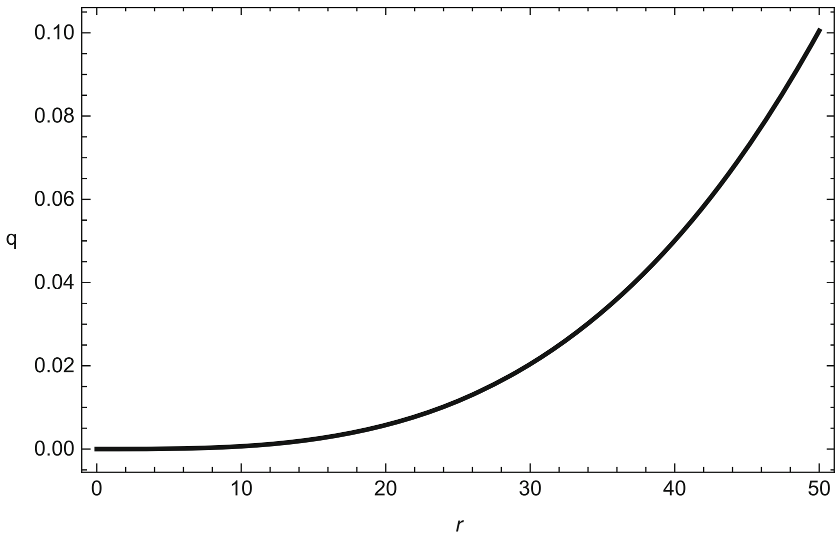

Fig. 3. Radial profile of the heat flux q.

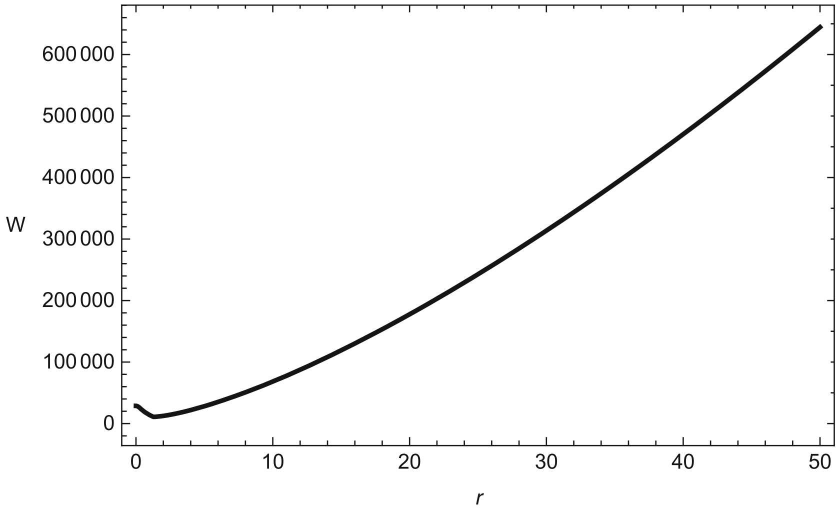

Fig. 4. Radial profile of the weak energy condition W.

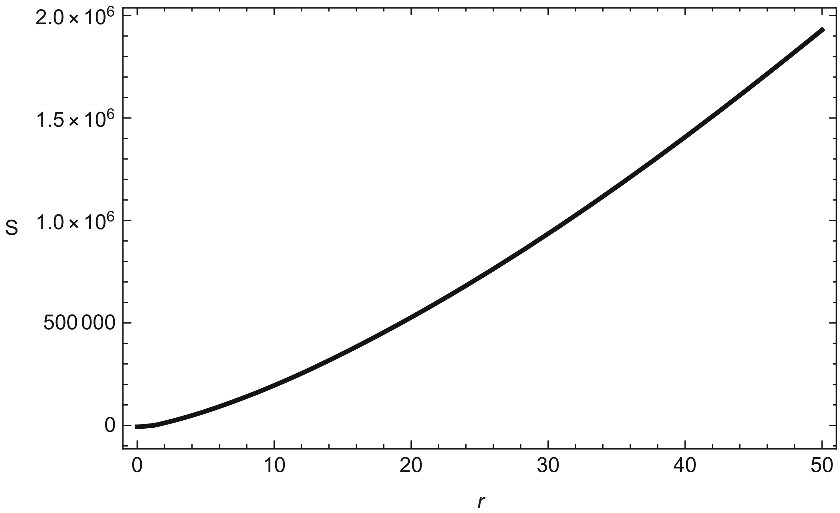

Fig. 5. Radial profile of the strong energy condition S.

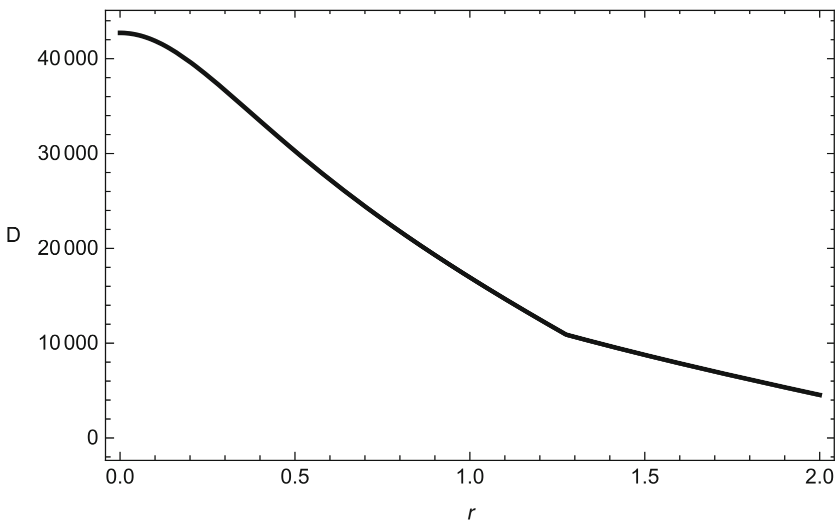

Fig. 6. Radial profile of the dominant energy condition D.

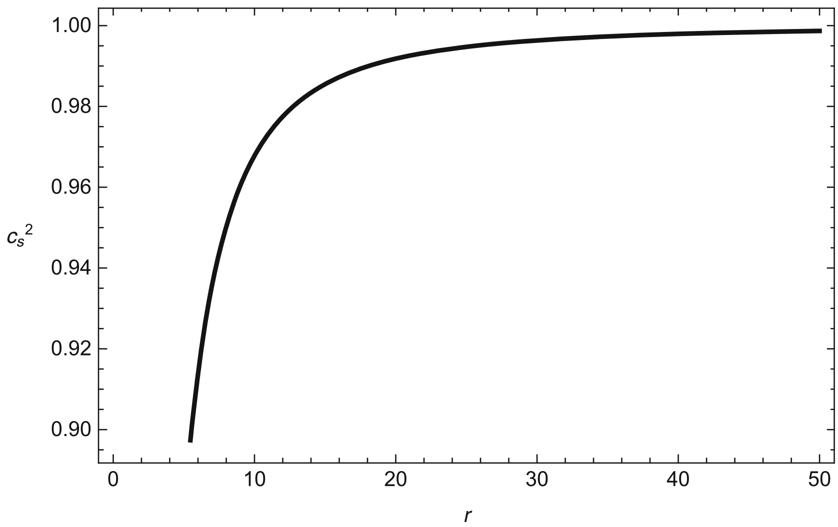

Fig. 7. Radial profile of the sound speed c_s^2.

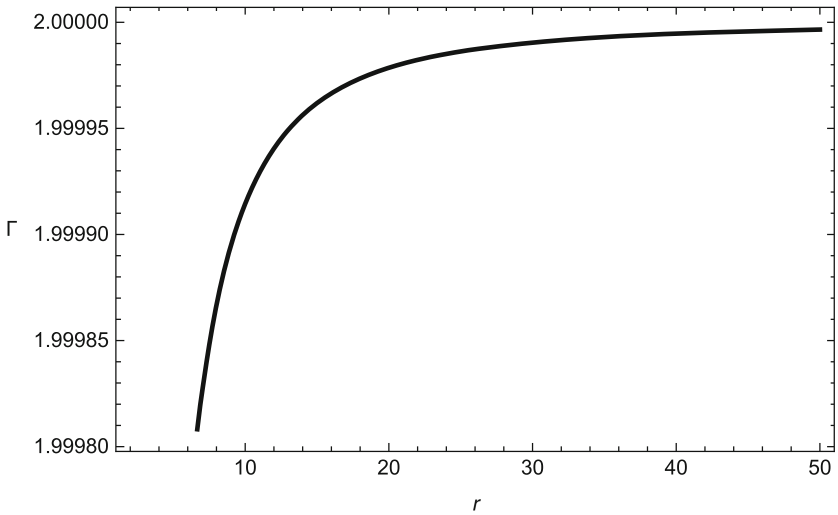

Fig. 8. Radial profile of the adiabatic index Γ.

5 Temperature evolution

The Eckart temperature equation is known to produce unrealistic behaviour for highly relativistic matter. This issue may be remedied with the application of the causal thermodynamics of Israel and Stewart [17]. The relativistic causal transport equation for the temperature, or the Maxwell-Cattaneo heat transport equation, has been developed in general by [22]. In the absence of rotation and viscous stresses the equation has the form

$$\tau h^a_b \dot{q}_a + q_b = -\kappa (h^a_b T_{;a} + T\dot{u}_b), \tag{19}$$

where $h_{ab} = g_{ab} + u_b u_a$ is the projection tensor, T is the temperature, κ (≥ 0) is the coefficient of thermal conductivity and τ is the relaxational time scale. It is the presence of τ that gives rise to causal and stable behaviour in the Israel-Stewart theory. When $\tau = 0$ we regain the Fourier transport equation.

Equation (19) is a truncated representation and insights into the values of τ and κ are required in order to solve it. Following the approach of Martínez [23], we assume

$$\kappa = \gamma T^3 \tau_c, \tag{20}$$

where

$$\tau_c = \left(\frac{\phi}{\gamma}\right) T^{-\sigma},$$

γ is a constant and $\tau_c = \frac{1}{n\sigma v}$ is the mean collision time (with σ being the collision cross section, v the mean particle speed and n is the total number of particles). We can make the following assumption:

$$\tau = \left(\frac{\beta\gamma}{\phi}\right)\tau_c = \beta T^{-\sigma}, \tag{21}$$

on physical grounds where ϕ, β and σ are nonnegative constants. Therefore for the spherically symmetric metric (1), the causal evolution equation (19) reduces to the following:

$$\beta T^{-\sigma}(qB)_{,t} + q(AB) = -\phi\left(\frac{T^{3-\sigma}(AT)_{,r}}{B}\right), \tag{22}$$

which is a nonlinear partial differential equation.

To integrate the transport equation (22), certain assumptions need to be made for the parameters β and σ. Following Govinder and Govender [24], we consider two cases: $\beta = 0$ which is the noncausal Eckart theory, and the causal case $\beta \neq 0$. We shall present analytical expressions for the temperatures resulting from the solution (12).

For the case $\beta = 0$, the noncausal solutions obtained by the integration of (22) are

$$(AT)^{4-\sigma} = \frac{\sigma - 4}{\phi}\int\left(A^{4-\sigma}qB^2\right)\mathrm{d}r + F(t), \quad \sigma \neq 4, \tag{23}$$

and

$$\ln(AT) = -\frac{1}{\phi}\int qB^2\mathrm{d}r + F(t), \quad \sigma = 4. \tag{24}$$

In both cases, $F(t)$ is an integration function and for the purpose of this analysis, we set it to zero. We will introduce the following nomenclature:

a) $T_{\mathrm{NC1}}(r,t)$ is the noncausal temperature resulting from (23).
b) $T_{\mathrm{NC2}}(r,t)$ is the noncausal temperature resulting from (24).

For solution (12), when $k = 2$, $c(t) = 1$, $d(t) = 0$, $a(t) = t$ and $b(t) = 1$, the analytical expressions for (23) and (24), respectively, are

$$T_{\mathrm{NC1}}(r,t) = \left(-\frac{1}{2}(2+\sqrt{2})^3(r^2+t)^{\frac{1}{2}\left(-1-\frac{3\sqrt{2}}{2}\right)}\right)^{\frac{1}{3}}, \tag{25a}$$

$$T_{\mathrm{NC2}}(r,t) = \frac{\exp\left[-\frac{1}{6}(-2+\sqrt{2})(2+\sqrt{2})\left[(r^2+t)^{\frac{1}{2}\left(-1-\frac{\sqrt{2}}{2}\right)}\right]^3\right]}{(r^2+t)^{\frac{\sqrt{2}}{2}}}. \tag{25b}$$

Hence we have explicit functional forms for the Eckart temperature for our model.

When the mean collision time vanishes and $\beta \geq 0$, eq. (22) can be integrated to give the following expression:

$$(AT)^4 = -\frac{4}{\phi}\left[\beta\int A^3B(qB)_{,t}\mathrm{d}r + \int A^4qB^2\mathrm{d}r\right] + F(t), \quad \sigma = 0, \tag{26}$$

where $F(t)$ is a function of integration. In the case when $0 \neq \sigma = 4$ the evolution equation (22) can be integrated by treating it as a Bernoulli equation in the variable (AT). We obtain the temperature

$$\begin{aligned}(AT)^4 = {} & \frac{-4\beta}{\phi}\exp\left(-\int\frac{4qB^2}{\phi}\mathrm{d}r\right)\int A^3B(qB)_{,t}\exp\left(\int\frac{4qB^2}{\phi}\mathrm{d}r\right)\\ & + F(t)\exp\left(-\int\frac{4qB^2}{\phi}\mathrm{d}r\right), \quad \sigma = 4.\end{aligned} \tag{27}$$

We utilise the following nomenclature:

a) $T_{\mathrm{C1}}(r,t)$ is the causal temperature resulting from (26).
b) $T_{\mathrm{C2}}(r,t)$ is the causal temperature resulting from (27).

For solution (12), when $F(t) = 0$, $\phi = -1$, $\beta = 1$, $k = 2$, $a(t) = t$, $c(t) = 1$, $d(t) = 0$ and $b(t) = 1$, eq. (26) becomes

$$T_{\mathrm{C}1}(r,t) = \frac{1}{(r^2+t)^{\frac{1}{\sqrt{2}}}} \left(-4 \left[\frac{27}{7}(r^2+t)^{\frac{1}{2}\left(-3+\frac{5\sqrt{2}}{2}\right)}\sqrt{2} + \frac{38}{7}(r^2+t)^{\frac{1}{2}\left(-3+\frac{5\sqrt{2}}{2}\right)} + \frac{113}{41}(r^2+t)^{\frac{1}{2}\left(-5+\frac{3\sqrt{2}}{2}\right)} + \frac{79}{41}(r^2+t)^{\frac{1}{2}\left(-5+\frac{3\sqrt{2}}{2}\right)}\sqrt{2} \right] \right)^{\frac{1}{4}}. \quad (28)$$

For the same quantities $F(t) = 0$, $\phi = -1$, $\beta = 1$, $k = 2$, $a(t) = t$, $c(t) = 1$, $d(t) = 0$ and $b(t) = 1$, (27) becomes

$$T_{\mathrm{C}2}(r,t) = \frac{1}{(r^2+t)^{\frac{\sqrt{2}}{2}}} \left(\exp\left[-\frac{2}{3}(2+\sqrt{2})^2 (r^2+t)^{\frac{1}{2}\left(3+\frac{3\sqrt{2}}{2}\right)}(2+\sqrt{2})(4+\sqrt{2}) \right] \times \int r(r^2+t)^{\frac{1}{2}\left(-7+\frac{3\sqrt{2}}{2}\right)} \exp\left[2r(r^2+t)^{-\frac{1}{2}\left(5+\frac{3\sqrt{2}}{2}\right)}(2+\sqrt{2})^2 \right] \mathrm{d}r \right)^{\frac{1}{4}}. \quad (29)$$

The causal temperatures (28) and (29) are given in closed form, the former expression is in terms of elementary functions and the latter expression has to be evaluated numerically.

6 Discussion

In this paper we studied a spherically symmetric shear-free spacetime and the associated model used to describe a relativistic heat conducting fluid distribution. We generated a new solution which is applicable to a stellar object in relativistic astrophysics. Our models has vanishing shear and describes heat flow in the interior of the fluid. The consistency condition arising from the isotropy of the fluid pressure was analysed and it was shown that the resulting master equation was a second-order ordinary differential equation with variable coefficients in the gravitational variables A and B. A particular form was chosen for one of the potentials and a new solution was obtained. A physical analysis of the solution was then undertaken and graphical profiles were generated for the energy density, pressure and heat flux, all of which showed a behaviour consistent with a realistic model. Plots were then generated for the energy conditions and it was shown that the weak, strong and dominant conditions were all satisfied, and that causality was not violated. Finally a plot for the effective adiabatic index was generated and its behaviour was positive, finite and decreasing as one moves radially outward through the fluid distribution. The thermal evolution of the fluid was then studied in detail. The Eckart [14] and Maxwell-Cattaneo heat transport equation [15] can be integrated. Expressions were found for the causal and noncausal cases relating to our heat conducting model.

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