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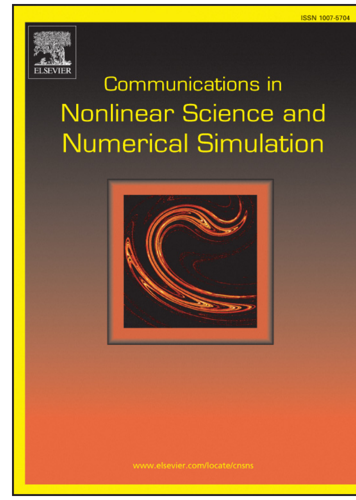

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Complete Group Classification of Systems of Two Linear Second-Order Ordinary Differential Equations

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Abstract

We give a complete group classification of the general case of linear systems of two second-order ordinary differential equations excluding the case of systems which are studied in the literature. This paper gives the initial step in the study of nonlinear systems of two second-order ordinary differential equations. It can also be extended to systems of equations with more than two equations. Furthermore the complete group classification of a system of two linear second-order ordinary differential equations is done. Four cases of linear systems of equations with inconstant coefficients are obtained.

Key words: Group classification, linear equations, admitted Lie group, equivalence transformation

1 Introduction

In this paper we consider the complete group classification of systems of two linear second-order ordinary differential equations. Systems of second-order ordinary differential equations are of great interest in the sciences and arise in many areas of physics, chemistry and mathematics. One of the main features

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of these differential equations is their symmetry properties. Many results on a scalar ordinary differential equation were obtained by the founder of symmetry analysis of differential equations. Lie [1] gave a complete group classification of a scalar ordinary differential equation of the form

$$y'' = f(x, y).$$

Later this classification was performed in a different way by L.V.Ovsiannikov [2]. This classification was obtained by directly solving the determining equations and exploiting the equivalence transformations. The same approach was applied in [3] for the group classification of more general type of equations of the form $y'' = P_3(x, y; y')$, where $P_3(x, y; y')$ is a polynomial of third degree with respect to the first-order derivative y'. In the general case of a scalar ordinary differential equation, $y'' = f(x, y, y')$, the application of the method that involves directly solving the determining equations gives rise to overwhelming difficulties. The group classification of such equations [4] is based on the enumeration of all possible Lie algebras of operators acting on the plane (x, y). Lie [1] gave the classification of all dissimilar Lie algebras (under complex change of variables) in two complex variables. In 1992, Gonzalez-Lopez et al. ordered the Lie classification of realizations of complex Lie algebras and extended it to the real case [5] [1]. A significant amount of results on the dimension and structure of symmetry algebras of linearizable ordinary differential equations is well known (see [4,6–11]).

Whereas symmetry properties of a scalar ordinary differential equation are well studied in the literature, the group classification of a system of even two linear second-order equations with constant coefficients is not complete. In recent works [11–15] the authors focused on the study of systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{1}$$

where M is a matrix with constant entries.

In the general case of systems of two linear second-order ordinary differential equations the more advanced results are obtained in [16], where using the canonical form,

$$\mathbf{y}'' = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \mathbf{y},$$

the authors presented several admitted Lie groups. It has been proven that a system of two linear second-order ordinary differential equations can have 5, 6, 7, 8 or 15 point symmetries. However the list of all distinguished represen-

[1] The mentioned works do not exhaust all papers.

tatives of systems of two linear second-order ordinary differential equations was not obtained in [16].

In this paper we give a complete group classification of the general case of linear systems of two second-order ordinary differential equations. The system considered here is a generalisation of Lie's study [1]. We exclude from our consideration the studied case (1) and the degenerate case given as follows:

$$y'' = F(x, y, z), \ z'' = 0. \tag{2}$$

The results found are new and have not been reported in the literature as far as we are aware.

The paper is organized as follows:

The first part of the paper deals with the preliminary study of systems of two second-order nonlinear equations of the form $y'' = F(x, y, z)$ and $z'' = G(x, y, z)$ using Ovsiannikov's approach [2]. This approach involves simplifying one generator and finding the associated functions. These functions are then used to solve the determining equations. Using this approach one can classify second-order nonlinear differential equations but this case has been left for a current project. The second part of the paper gives a complete treatment of systems of second-order linear equations. This is then followed by the results, discussion and conclusion.

2 Preliminary study of systems of nonlinear equations

A system of two second-order nonlinear differential equations of the form

$$y'' = F(x, y, z), \quad z'' = G(x, y, z) \tag{3}$$

is considered in this section. In matrix form equations (3) are given by

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}),$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F(x, y, z) \\ G(x, y, z) \end{pmatrix}.$$

We consider the system of nonlinear equations here as it will later allow us to separate equations given into their respective classes. We exclude from the study the trivial systems where one of the equations can be reduced to the system with $G = 0$. From these classes we will select linear systems of equations. For this purpose we use equivalence transformations and the respective solutions of the determinining equations.

2.1 Equivalence transformations

Calculations show that the equivalence Lie group is defined by the generators:

$$X_1^e = y\partial_y + F\partial_F, \quad X_2^e = z\partial_y + G\partial_F, \quad X_3^e = y\partial_z + F\partial_G, \quad X_4^e = z\partial_z + G\partial_G,$$
$$X_5^e = \phi_1(x)\partial_y + \phi_1''(x)\partial_F, \quad X_6^e = \phi_2(x)\partial_z + \phi_2''(x)\partial_G,$$
$$X_7^e = 2\xi(x)\partial_x + \xi'(x)y\partial_y + \xi'(x)z\partial_z + (\xi'''(x)y - 3\xi'(x)F)\partial_F + (\xi'''(x)z - 3\xi'(x)G)\partial_G,$$

where $\xi(x)$, $\phi_1(x)$ and $\phi_2(x)$ are arbitrary functions.

The transformations related with the generators X_1^e, X_2^e, X_3^e and X_4^e correspond to the linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant nosingular matrix P. The transformations corresponding to the generators X_5^e and X_6^e define the change

$$\tilde{y} = y + \varphi_1(x), \quad \tilde{z} = z + \varphi_2(x).$$

The equivalence transformation related with the generator X_7^e is

$$\tilde{x} = \varphi(x), \ \tilde{y} = y\psi(x), \ \tilde{z} = z\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}. \tag{4}$$

Equation (4) also appears in the group classifcation of a single equation of the same form [2].

2.2 Determining equations

Consider the generator

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta_1(x, y, z)\frac{\partial}{\partial y} + \eta_2(x, y, z)\frac{\partial}{\partial z}.$$

According to the Lie algorithm [9], X is admitted by system (3) if it satisfies the associated determining equations. The first part of the determining equations is

$$3\xi_1(F_y - G_z) + \xi_2 G_y = 0, \ \ \xi_1 G_y = 0,$$

$$\xi_1 F_z + 3\xi_2(-F_y + G_z) = 0, \ \ \xi_2 F_z = 0,$$

where

$$\xi(x, y, z) = \xi_1(x)y + \xi_2(x)z + \xi_0(x).$$

From these equations one can conclude that $\xi_1^2 + \xi_2^2 \neq 0$ only for the case where

$$F_y - G_z = 0, \quad F_z = 0, \quad G_y = 0. \tag{5}$$

Solving the last conditions (5)leads to the degenerate case

$$F(x, y, z) = a(x)y + b(x), \ \ G(x, y, z) = a(x)z + h(x).$$

Using a particular solution and equivalent transformations, equations (3) are reduced to the trivial case of the free particle equation.

We consider the case where the conditions (5) are not satisfied. In this case

$$\xi_1 = 0, \ \ \xi_2 = 0$$

and the determining equations are given by

$$F_z(\xi' z + zk_4 + yk_3 + \zeta_2) + F_y(\xi' y + zk_2 + yk_1 + \zeta_1)$$
$$+2F_x\xi - \xi''' y + 3\xi' F - \zeta_1'' - k_1 F - k_2 G = 0,$$

$$G_z(\xi' z + zk_4 + yk_3 + \zeta_2) + G_y(\xi' y + zk_2 + yk_1 + \zeta_1)$$
$$+2G_x\xi - \xi''' z + 3\xi' G - \zeta_2'' - k_3 F - k_4 G = 0,$$

where an admitted generator has the form

$$X = 2\xi(x)\partial_x + (y\xi'(x) + k_1 y + k_2 z + \zeta_1(x))\partial_y + (\xi' z + k_3 y + k_4 z + \zeta_2(x))\partial_z,$$

and k_i, $(i = 1, 2, ...)$ are constant.

We further separate the study of the determining equations into two different cases:

(a) the case in which there is at least one admitted generator with $\xi \neq 0$;

(b) the case in which for all admitted generators $\xi = 0$.

2.3 Case $\xi \neq 0$

We consider the generator X_o for which $\xi \neq 0$. Using the equivalence transformation:

$$y_1 = y + \varphi(x), \quad z_1 = z + \psi(x),$$

the generator X_o becomes

$$\begin{aligned} X_o = 2\xi\partial_x &+ (\xi' y_1 + z_1 k_2 + y_1 k_1 + 2\xi\varphi' - \xi'\varphi - \psi k_2 - \varphi k_1 + \zeta_1)\partial_{y_1} \\ &+(\xi' z_1 + z_1 k_4 + y_1 k_3 + 2\xi\psi' - \xi'\psi - \psi k_4 - \varphi k_3 + \zeta_2)\partial_{z_1}. \end{aligned}$$

One can choose the functions $\varphi(x)$ and $\psi(x)$ such that

$$\begin{aligned} &2\xi\varphi' - \xi'\varphi - \psi k_2 - \varphi k_1 + \zeta_1 = 0, \\ &2\xi\psi' - \xi'\psi - \psi k_4 - \varphi k_3 + \zeta_2 = 0. \end{aligned}$$

Then the generator X_o is reduced to

$$X_o = 2\xi\partial_x + (\xi' y + k_1 y + k_2 z)\partial_y + (\xi' z + k_3 y + k_4 z)\partial_z.$$

The equivalence transformation

$$x_1 = \alpha(x), \quad y_1 = y\beta(x), \quad z_1 = z\beta(x),$$

where

$$\alpha''\beta = 2\alpha'\beta', \quad (\alpha'\beta \neq 0),$$

reduces the generator X_o to

$$X_o = 2\alpha'\xi\partial_{x_1} + ((2\xi\beta'/\beta + \xi' + k_1)y_1 + z_1 k_2)\partial_{y_1} + (k_3 y_1 + (2\xi\beta'/\beta + \xi' + k_4)z_1)\partial_{z_1}.$$

Choosing $\beta(x)$ such that $2\xi\beta'/\beta + \xi' = 0$ the generator X_o is reduced to the generator

$$X_o = 2\alpha'\xi\partial_{x_1} + (k_1 y_1 + k_2 z_1)\partial_{y_1} + (k_3 y_1 + k_4 z_1)\partial_{z_1}.$$

Notice that in this case

$$\frac{d(\alpha'\xi)}{dx_1} = 0.$$

Indeed

$$\frac{d(\alpha'\xi)}{dx_1} = \frac{(\alpha'\xi)'}{\alpha'} = \xi' + \frac{\alpha''}{\alpha'}\xi = -2\xi\frac{\beta'}{\beta} + 2\frac{\beta'}{\beta}\xi = 0.$$

Thus the generator X_o is

$$X_o = k\partial_{x_1} + (k_1 y_1 + k_2 z_1)\partial_{y_1} + (k_3 y_1 + k_4 z_1)\partial_{z_1},$$

where $k = 2\alpha'\xi \neq 0$ is constant. We rewrite the generator X_o in the form

$$X_o = \partial_x + (a_{11}y + a_{12}z)\partial_y + (a_{21}y + a_{22}z)\partial_z.$$

The determining equations become

$$\begin{aligned}
&(a_{11}y + a_{12}z)F_y + (a_{21}y + a_{22}z)F_z + F_x - a_{11}F - a_{12}G = 0,\\
&(a_{11}y + a_{12}z)G_y + (a_{21}y + a_{22}z)G_z + G_x + a_{21}F - a_{22}G = 0.
\end{aligned} \tag{6}$$

Here a_{ij}, $(i, j = 1, 2)$ are constant. In the matrix form these equations are rewritten as

$$\left((A\mathbf{y})^t\nabla\right)\mathbf{F} + \mathbf{F}_x - A\mathbf{F} = 0, \tag{7}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ \nabla = \begin{pmatrix} \partial_y \\ \partial_z \end{pmatrix}.$$

Further simplifications are related with simplifications of the matrix A.

We apply the change $\tilde{\mathbf{y}} = P\mathbf{y}$ where P is a nonsingular matrix with constant entries

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Equations (3) become

$$\tilde{\mathbf{y}}'' = \widetilde{\mathbf{F}}(x, \tilde{\mathbf{y}})$$

where

$$\widetilde{\mathbf{F}}(x, \tilde{\mathbf{y}}) = P\mathbf{F}(x, P^{-1}\tilde{\mathbf{y}}).$$

The partial derivatives ∂_y and ∂_z are changed as follows

$$\nabla = P^t\widetilde{\nabla}.$$

Hence equations (7) are changed as

$$\begin{aligned}
\left((AP^{-1}\tilde{\mathbf{y}})^tP^t\widetilde{\nabla}\right)(P^{-1}\widetilde{\mathbf{F}}) + P^{-1}\widetilde{\mathbf{F}}_x - AP^{-1}\widetilde{\mathbf{F}} &= P^{-1}\left(\left((PAP^{-1}\tilde{\mathbf{y}})^t\widetilde{\nabla}\right)\widetilde{\mathbf{F}} + \widetilde{\mathbf{F}}_x - PAP^{-1}\widetilde{\mathbf{F}}\right)\\
&= P^{-1}\left(\left((\widetilde{A}\tilde{\mathbf{y}})^t\widetilde{\nabla}\right)\widetilde{\mathbf{F}} + \widetilde{\mathbf{F}}_x - \widetilde{A}\widetilde{\mathbf{F}}\right) = 0,
\end{aligned}$$

where

$$\widetilde{A} = PAP^{-1}.$$

This means that the change $\tilde{\mathbf{y}} = P\mathbf{y}$ reduces equation (7) to the same form with the matrix A changed. The infinitesimal generator is also changed as

$$X_o = \partial_x + (\widetilde{A}\tilde{\mathbf{y}})^t\widetilde{\nabla}.$$

Using this change, the matrix A can be presented in the Jordan form. For a real-valued 2×2 matrix A, if the matrix P also has real-valued entries, then the Jordan matrix is one of the following three types:

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \tag{8}$$

where a, b and $c > 0$ are real numbers. Notice also that using the dilation of x, one can reduce c to 1.

2.3.1 Case $A = J_1$

We assume that

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

In this case the equations for the functions F and G are

$$ayF_y + bzF_z + F_x = aF,$$
$$ayG_y + bzG_z + G_x = bG.$$

The general solution of these equations is

$$F(x,u,v) = e^{ax} f(u,v), \quad G(x,u,v) = e^{bx} g(u,v) \tag{9}$$

where

$$u = ye^{-ax}, \quad v = ze^{-bx}.$$

The admitted generator is

$$X_o = \partial_x + ay\partial_y + bz\partial_z.$$

2.3.2 Case $A = J_2$

We assume that

$$A = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}.$$

In this case equations (6) become

$$\begin{aligned} (ay + cz)F_y + (-cy + az)F_z + F_x - aF - cG = 0, \\ (ay + cz)G_y + (-cy + az)G_z + G_x + cF - aG = 0. \end{aligned} \tag{10}$$

Introducing the variables

$$u = e^{-ax}(y\cos(cx) - z\sin(cx)), \quad v = e^{-ax}(y\sin(cx) + z\cos(cx)),$$

equations (10) become

$$F_x - aF - cG = 0, \quad G_x + cF - aG = 0.$$

The general solution of these equations is

$$\begin{aligned} F(x,u,v) &= e^{ax}\left(\cos(cx)f(u,v) + \sin(cx)g(u,v)\right), \\ G(x,y,z) &= e^{ax}\left(-\sin(cx)f(u,v) + \cos(cx)g(u,v)\right) \end{aligned} \tag{11}$$

where $f(u,v)$ and $g(u,v)$ are arbitrary functions. The admitted generator is

$$X_o = \partial_x + (ay + cz)\partial_y + (-cy + az)\partial_z.$$

2.3.3 Case $A = J_3$

We assume that

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

In this case equations (6) become

$$\begin{aligned} (ay + z)F_y + azF_z + F_x - aF - G = 0, \\ (ay + z)G_y + azG_z + G_x - aG = 0. \end{aligned} \tag{12}$$

As in the previous case we introduce the variables

$$u = e^{-ax}(y - zx), \quad v = e^{-ax}z$$

so that equations (12) become

$$F_x - aF - G = 0, \quad G_x - aG = 0.$$

The general solution of these equations is

$$F(x,u,v) = e^{ax}\left(f(u,v) + xg(u,v)\right), \quad G(x,y,z) = e^{ax}g(u,v), \tag{13}$$

where $f(u,v)$ and $g(u,v)$ are arbitrary functions. The admitted generator is

$$X_o = \partial_x + (ay + z)\partial_y + az\partial_z.$$

2.4 Case $\xi = 0$

Substituting $\xi = 0$ into the determining equations, one finds that

$$(a_{11}y + a_{12}z + \zeta_1)F_y + (a_{21}y + a_{22}z + \zeta_2)F_z = a_{11}F + a_{12}G + \zeta_1'',$$
$$(a_{11}y + a_{12}z + \zeta_1))G_y + (a_{21}y + a_{22}z + \zeta_2)G_z = a_{21}F + a_{22}G + \zeta_2'', \tag{14}$$

or in a matrix form

$$\left((A\mathbf{y})^t\nabla + \mathbf{h}\right)\mathbf{F} = A\mathbf{F} + \mathbf{h}''$$

where

$$A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}, \quad \mathbf{h}(x) = \begin{pmatrix} \zeta_1(x) \\ \zeta_2(x) \end{pmatrix}.$$

Similarly to the case where $\xi \neq 0$ we use the Jordan forms (8) of the matrix A. The admitted generator has the form

$$X_o = (k_1y + k_2z + \zeta_1(x))\partial_y + (k_3y + k_4z + \zeta_1(x))\partial_z.$$

2.4.1 Case $A = J_1$

Assuming that $A = J_1$, equations (14) for the functions F and G are

$$(ay + h_1)F_y + (bz + h_2)F_z = aF + h_1'',$$
$$(ay + h_1)G_y + (bz + h_2)G_z = bG + h_2''.$$

The general solution of these equations depends on a value of a and b:

- Case: $a \neq 0$, $b \neq 0$

$$aF + h_1'' = (ay + h_1)f(x, v), \quad bG + h_2'' = (bz + h_2)g(x, v),$$
$$v = (bz + h_2)^a(ay + h_1)^{-b}.$$

- $a \neq 0$, $b = 0$

$$aF + h_1'' = (ay + h_1)f(x, v), \quad G = \tfrac{h_2''}{a}\ln(ay + h_1) + g(x, v),$$
$$v = z - \tfrac{h_2}{a}\ln(ay + h_1).$$

- $a = 0$, $b = 0$, $h_1 \neq 0$

$$F = \tfrac{h_1''}{h_1}y + f(x, v), \quad G = \tfrac{h_2''}{h_1}y + g(x, v),$$
$$v = z - \tfrac{h_2}{h_1}y.$$

Here $f(x,v)$ and $g(x,v)$ are arbitrary functions.

2.5 Case $A = J_2$

In this case equations (14) become

$$\begin{aligned}(ay + cz + h_1)F_y + (-cy + az + h_2)F_z &= aF + cG + h_1'',\\ (ay + cz + h_1)G_y + (-cy + az + h_2)G_z &= -cF + aG + h_2''.\end{aligned} \tag{15}$$

Introducing the variables

$$y = \overline{y} - (a^2 + c^2)^{-1}(ah_1 - ch_2), z = \overline{z} - (a^2 + c^2)^{-1}(ch_1 + ah_2),$$

$$F = \overline{F} - (a^2 + c^2)^{-1}(ah_1'' - ch_2''), \quad G = \overline{G} - (a^2 + c^2)^{-1}(ch_1'' + ah_2'')$$

equations (15) become

$$\begin{aligned}(a\overline{y} + c\overline{z})\overline{F}_{\overline{y}} + (-c\overline{y} + a\overline{z})\overline{F}_{\overline{z}} &= a\overline{F} + c\overline{G},\\ (a\overline{y} + c\overline{z})\overline{G}_{\overline{y}} + (-c\overline{y} + a\overline{z})\overline{G}_{\overline{z}} &= -c\overline{F} + a\overline{G}.\end{aligned}$$

In the variables

$$\overline{y} = ve^{au}\sin(cu), \quad \overline{z} = ve^{au}\cos(cu),$$

these equations are

$$\begin{aligned}\overline{F}_u &= a\overline{F} + c\overline{G},\\ \overline{G}_u &= -c\overline{F} + a\overline{G}.\end{aligned}$$

The general solution of the last equations is

$$\overline{F}(x,u,v) = e^{au}\left(\cos(cu)f(x,v) + \sin(cu)g(x,v)\right),$$

$$\overline{G}(x,u,v) = e^{au}\left(-\sin(cu)f(x,v) + \cos(cu)g(x,v)\right)$$

where $f(x,v)$ and $g(x,v)$ are arbitrary functions.

2.5.1 Case $A = J_3$

In this case equations (14) become

$$\begin{aligned}(ay + z + h_1)F_y + (az + h_2)F_z &= aF + G + h_1'',\\ (ay + z + h_1)G_y + (az + h_2)G_z &= aG + h_2''.\end{aligned} \tag{16}$$

The general solution of the equations depends on a and h_2. If we consider the case for which $a = 0$, then similarly to the previous case, we introduce the variables

$$z = \overline{z} - h_1, \quad G = \overline{G} - h_1''$$

so that equations (15) become

$$\begin{aligned}
\overline{z}\overline{F}_y + h_2\overline{F}_{\overline{z}} &= \overline{G}, \\
\overline{z}\overline{G}_y + h_2\overline{G}_{\overline{z}} &= h_2''.
\end{aligned}$$

If $h_2 = 0$, then

$$F(x, y, z) = \tfrac{y}{z+h_1} f(x, v) + g(x, v),$$

$$G(x, y, z) = -h_1'' + f(x, v)$$

where $v = z + h_1$.

If $h_2 \neq 0$, then

$$\begin{aligned}
F(x, y, z) &= \tfrac{h_2''}{2} u^2 + u f(x, v) + g(x, v), \\
G(x, y, z) &= -h_1'' + h_2'' u + f(x, v)
\end{aligned}$$

where $u = \frac{z+h_1}{h_2}$, $v = y - \frac{(z+h_1)^2}{2h_2}$.

If $a \neq 0$ we introduce the variables

$$y = \overline{y} - a^{-2}(ah_1 - h_2), z = \overline{z} - a^{-1}h_2,$$

$$F = \overline{F} - a^{-2}(ah_1'' - h_2''), \quad G = \overline{G} - a^{-1}h_2''.$$

Equation (15) becomes

$$\begin{aligned}
(a\overline{y} + \overline{z})\overline{F}_{\overline{y}} + a\overline{z}\overline{F}_{\overline{z}} &= a\overline{F} + \overline{G}, \\
(a\overline{y} + \overline{z})\overline{G}_{\overline{y}} + a\overline{z}\overline{G}_{\overline{z}} &= a\overline{G}.
\end{aligned}$$

In the variables

$$\overline{y} = uve^{au}, \quad \overline{z} = ve^{au}$$

the equations are

$$\begin{aligned}
\overline{F}_u &= a\overline{F} + \overline{G}, \\
\overline{G}_u &= a\overline{G}.
\end{aligned}$$

The general solution of these equations is

$$\begin{aligned}
\overline{F}(x, u, v) &= e^{au}\left(uf(x, v) + g(x, v)\right), \\
\overline{G}(x, u, v) &= e^{au} f(x, v)
\end{aligned}$$

where $f(x, v)$ and $g(x, v)$ are arbitrary functions.

3 Systems of linear equations

Linear second-order ordinary differential equations have the following form,

$$\mathbf{y}'' = A(x)\mathbf{y}' + B(x)\mathbf{y} + f(x), \tag{17}$$

where $A(x)$ and $B(x)$ are matrices, and $f(x)$ is a vector. Using a particular solution $\mathbf{y}_p(x)$ and the change

$$\mathbf{y} = \tilde{\mathbf{y}} + \mathbf{y}_p,$$

we can, without loss of generality assume that $f(x) = 0$. The matrix $A(x)$ or $B(x)$ can also be assumed to be zero if we use the change, $\mathbf{y} = C(x)\tilde{\mathbf{y}}$, where $C = C(t)$ is a nonsingular matrix. In the present paper the matrix $A(x)$ is reduced to zero. In this case the linear equations (3) are linear functions of y and z:

$$F(x, y, z) = c_{11}(x)y + c_{12}(x)z, \quad G(x, y, z) = c_{21}(x)y + c_{22}(x)z. \tag{18}$$

Any linear system of equations admits the following generators

$$y\partial_y + z\partial_z, \tag{19}$$

$$\zeta_1(x)\partial_x,\ \zeta_2(x)\partial_x \tag{20}$$

where $\zeta_1(x)$ and $\zeta_2(x)$ are solutions of the equations:

$$\zeta_1'' = c_{11}\zeta_1 + c_{12}\zeta_2,\ \zeta_2'' = c_{21}\zeta_1 + c_{22}\zeta_2.$$

For the classification problem one needs to study equations which admit generators different from (19) and (20).

3.1 Equivalence transformations

Calculations show that the equivalence Lie group is defined by the generators:

$$X_1^e = z\partial_y + c_{21}\partial_{c_{11}} + (c_{22} - c_{11})\partial_{c_{12}} - c_{21}\partial_{c_{22}},$$
$$X_2^e = y\partial_z - c_{12}\partial_{c_{11}} - (c_{22} - c_{11})\partial_{c_{21}} + c_{12}\partial_{c_{22}},$$

$$X_3^e = y\partial_y + z\partial_z, \quad X_4^e = y\partial_y - z\partial_z + 2(c_{12}\partial_{c_{12}} - c_{21}\partial_{c_{21}}),$$

$$X_5^e = 2\xi(x)\partial_x + \xi'(x)y\partial_y + \xi'(x)z\partial_z + (\xi'''(x) - 4\xi'(x)c_{11})\partial_{c_{11}}$$
$$-4\xi'(x)c_{12}\partial_{c_{12}} - 4\xi'(x)c_{21}\partial_{c_{21}} + (\xi'''(x) - 4\xi'(x)c_{22})\partial_{c_{22}},$$

where $\xi(x)$ is an arbitrary function.

As is similar to nonlinear equations the transformations related with the generators X_1^e, X_2^e, X_3^e and X_4^e correspond to the linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant nosingular matrix P. The transformations corresponding to the generators X_5^e is

$$\widetilde{x} = \varphi(x),\ \widetilde{y} = y\psi(x),\ \widetilde{z} = z\psi(x)$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

3.2 Case $\xi \neq 0$

Using the obtained general forms of equations admitting an infinitesimal generator with $\xi \neq 0$ linear systems of equations (9), (11) and (13) have the following form:

$$f(u,v) = \alpha_{11}u + \alpha_{12}v, \quad g(u,v) = \alpha_{21}u + \alpha_{22}v. \tag{21}$$

3.2.1 Case $A = J_1$

The functions F and G become

$$F = \alpha_{11}y + e^{\alpha x}\alpha_{12}z, \ \ G = e^{-\alpha x}\alpha_{21}y + \alpha_{22}z$$

where $\alpha = a - b$.

Notice that if $\alpha_{12} = \alpha_{21} = 0$, then the linear system of equations is a linear system with constant coefficients. This case has been studied [12,13]. For the same reason, one assumes that $\alpha \neq 0$. Hence, without loss of generality one can assume that $\alpha\alpha_{12} \neq 0$. Using the dilation of x and then z, one can set $\alpha = \alpha_{12} = 1$. Thus,

$$F = \alpha_{11}y + e^{x}z, \ \ G = e^{-x}\alpha_{21}y + \alpha_{22}z. \tag{22}$$

Since for $\alpha_{21} = 0$ the equations are reduced to the case where $G = 0$. One also has to assume that $\alpha_{21} \neq 0$.

3.2.2 Case $A = J_2$

Substituting (21) into the functions F and G one finds

$$\begin{aligned} F &= y(\cos^2(cx)\alpha_{11} + \cos(cx)\sin(cx)\alpha_{12} + \cos(cx)\sin(cx)\alpha_{21} + \sin^2(cx)\alpha_{22}) \\ &\quad + z(\cos^2(cx)\alpha_{12} - \cos(cx)\sin(cx)\alpha_{11} + \cos(cx)\sin(cx)\alpha_{22} - \sin^2(cx)\alpha_{21}), \\ G &= y(\cos^2(cx)\alpha_{21} - \cos(cx)\sin(cx)\alpha_{11} + \cos(cx)\sin(cx)\alpha_{22} - \sin^2(cx)\alpha_{12}) \\ &\quad + z(\cos^2(cx)\alpha_{22} - \cos(cx)\sin(cx)\alpha_{12} - \cos(cx)\sin(cx)\alpha_{21} + \sin^2(cx)\alpha_{11}). \end{aligned}$$

Using trigonometry formulae, and introducing the constants α, β, c_1 and c_2:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \beta + c_2 & \alpha - c_1 \\ \alpha + c_1 & -\beta + c_2 \end{pmatrix},$$

one can rewrite functions F and G in the form

$$\begin{aligned} F &= y(\cos(2cx)\beta + \sin(2cx)\alpha + c_2) + z(\cos(2cx)\alpha - \sin(2cx)\beta - c_1), \\ G &= y(\cos(2cx)\alpha - \sin(2cx)\beta + c_1) - z(\cos(2cx)\beta + \sin(2cx)\alpha - c_2). \end{aligned}$$

Without loss of generality one can assume that $\beta = 0$. If $\beta \neq 0$, then the change $\tilde{\mathbf{y}} = P\mathbf{y}$ with the matrix

$$P = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

and the angle θ satisfying the equation,

$$\beta\tau^4 + 4\alpha\tau^3 - 6\beta\tau^2 - 4\alpha\tau + \beta = 0,$$

reduces the functions F and G to the form

$$\begin{aligned} F &= y(\gamma\sin(2cx) + c_2) + z(\gamma\cos(2cx) - c_1), \\ G &= y(\gamma\cos(2cx) + c_1) + z(-\gamma\sin(2cx) + c_2). \end{aligned}$$

Here $\tau = \tan\theta$. Since for $\gamma = 0$ the linear system of equations is a linear system with constant coefficients one has to consider $\gamma \neq 0$. Without loss of generality one can set $\gamma = 2c = 1$:

$$\begin{aligned} F &= y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G &= y(\cos(x) + c_1) + z(-\sin(x) + c_2). \end{aligned}$$

3.2.3 Case $A = J_3$

Substituting (21) into the functions (18) one finds

$$F = y(\alpha_{11} + \alpha_{21}x) + z(\alpha_{12} + (\alpha_{22} - \alpha_{11})x - \alpha_{21}x^2),$$
$$G = y\alpha_{21} + z(-\alpha_{21}x + \alpha_{22}). \tag{23}$$

In particular for $\alpha_{21} = 0$ one has $G = z\alpha_{22}$. Using an equivalence transformation one can reduce $\alpha_{22} = 0$:

$$F = (y - zx)\alpha_{11} + z\alpha_{12}, \; G = 0. \tag{24}$$

This corresponds to the case which is omitted from the study. Thus one has to assume that $\alpha_{21} \neq 0$. Without loss of generality one can set $\alpha_{21} = 1$:

$$F = y(\alpha_{11} + x) + z(\alpha_{12} + (\alpha_{22} - \alpha_{11})x - x^2),$$
$$G = y + z(-x + \alpha_{22}). \tag{25}$$

3.3 Case $\xi = 0$

Substituting (18) into (14) and splitting it with respect to y and z one has

$$a_{21}c_{12} - a_{12}c_{21} = 0, \; a_{12}(c_{11} - c_{22}) + (a_{22} - a_{11})c_{12} = 0,$$
$$(a_{11} - a_{22})c_{21} + a_{21}(c_{22} - c_{11}) = 0, \tag{26}$$

$$\zeta_1'' = \zeta_1 c_{11} + \zeta_2 c_{12}, \; \zeta_2'' = \zeta_1 c_{21} + \zeta_2 c_{22}. \tag{27}$$

The admitted generator is of the form

$$X_o = (a_{11}y + a_{12}z + \zeta_1(x))\partial_y + (a_{21}y + a_{22}z + \zeta_1(x))\partial_z.$$

Equations (27) define the trivial set of the admitted generators (20). The nontrivial generators

$$X_o = (a_{11}y + a_{12}z)\partial_y + (a_{21}y + a_{22}z)\partial_z \tag{28}$$

are defined by equations (26). Equations (26) can be simplified by using the Jordan form of the matrix A.

If $A = J_1$ then equations (26) become

$$(b - a)c_{12} = 0, \; (b - a)c_{21} = 0.$$

Since for $b = a$ the generator (28) is also trivial, one has to assume that $b \neq a$. The last condition gives

$$c_{12} = 0,\ c_{21} = 0.$$

In this case the linear system of equations is reduced to the degenerated case $G = 0$. Similar reduction occurs for $A = J_3$. Indeed, if $A = J_3$, then equations (26) are

$$c_{21} = 0,\ c_{22} = c_{11}.$$

If $A = J_2$ then equations (26) give

$$c_{12} + c_{21} = 0,\ c_{22} = c_{11}.$$

Here one has to assume that $c_{12} \neq 0$. Using the equivalence transformation of the form

$$t = \varphi(x),\ u = y\psi(x),\ v = z\psi(x),$$

where $\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}$, one can reduce $c_{12} = 1$. Thus, in this case one has to also assume that $c_{11}' \neq 0$.

4 Solutions of the determining equations

The preliminary study has led us to the following linear systems which admit generators different from (19):

$$F = \alpha_{11}y + e^{x}z,\ \ G = e^{-x}\alpha_{21}y + \alpha_{22}z, s \tag{29}$$

$$\begin{gathered} F = y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G = y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{gathered} \tag{30}$$

$$\begin{gathered} F = y(\alpha_{11} + x) + z(\alpha_{12} + (\alpha_{22} - \alpha_{11})x - x^2), \\ G = y + z(-x + \alpha_{22}), \end{gathered} \tag{31}$$

$$F = yc + z, G = -y + zc \tag{32}$$

where $\alpha_{ij} = \alpha_{ij}$ and $c_i = c_i$ $(i, j = 1, 2)$ are constant, $c = c(x)$ and $\alpha_{21}c' \neq 0$.

4.1 Admitted Generators

Cases	Determining Equations	Admitted Generator
(29)	(3) with (29)	$\partial_x - z\partial_z$
(30)	(3) with (30)	$2\partial_x + z\partial_y - y\partial_z$
(31)	(3) with (30)	$\partial_x + z\partial_y$
(32)	(3) with (30)	$z\partial_y - y\partial_z$.

5 Discussion

Here we give comparisons of our study with results found in [16]. Since the complete comparison is very cumbersome we only do it for the case where in [16] there is two or more admitted generators. We further show that these cases are either equivalent to degenerate cases with the trivial second equation ($G(x, y, z) = 0$) or the case which is equivalent to the constant matrix A.

Indeed, the case (I.4.2) of [16] corresponds to the matrix

$$A = \begin{pmatrix} 0 & b(x) \\ 0 & 0 \end{pmatrix}$$

For this case $G(x, y, z) = 0$.

The case (I.6.5) of [16] corresponds to the matrix

$$A = \begin{pmatrix} 0 & b(x) \\ \gamma b(x) & 0 \end{pmatrix}$$

with a particular function

$$b(x) = b_0 g^{-1}(x)$$

where $g(x) = (c_1x^2/2 + c_2x + c_3) \neq 0$ and γ is constant. The admitted generators found in [16] are

$$X_6 = z\partial_y + \gamma y\partial_z, \quad X_7 = 2g(x)\partial_x + g'(x)(y\partial_y + z\partial_z).$$

In this case the change

$$x_1 = \alpha(x), \quad y_1 = y\beta(x), \quad z_1 = z\beta(x),$$

where

$$2g\beta'/\beta + g' = 0, \;\; \alpha''\beta = 2\alpha'\beta' \quad (\alpha'\beta \neq 0),$$

reduces the generator X_7 to

$$X_7 = \partial_{x_1},$$

which means that this case is equivalent to the case of two linear equations with constant coefficients.

Similarly one can show that the cases (I.8.2) and (II.3.3) of [16] are equivalent to linear systems with constant coefficients.

6 Conclusion

We have given a complete group classification of the general case of linear systems of two second-order ordinary differential equations excluding the case of systems which are equivalent to systems of type (1) and the degenerate case (2). As a starting point we gave a preliminary study of systems of two second-order nonlinear equations of the form $y'' = F(x, y, z)$ and $z'' = G(x, y, z)$ using Ovsiannikov's approach [2]. This involved simplifying one generator and finding the associated functions which were used to solve the determining equations. The study here can be considered as the first step in the study of nonlinear systems of two second-order ordinary differential equations and can be applied to systems of equations with more than two equations.

A comparison of our study with results found in the literature [16] is given in the Section on the Discussion.

The complete group classification of a system of two linear second-order ordinary differential equations was done. We found four cases of linear systems of equations which are not equivalent to the system (1) and the degenerate case.

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Highlights

- A complete group classification of the general case of linear systems of two second-order ordinary differential equations is given.
- Four cases of linear systems of equations with inconstant coefficients are obtained.
- A preliminary study of systems of two second-order nonlinear equations of the form $y'' = F(x, y, z),\ z'' = G(x, y, z)$ is given.