Geophysical & Astrophysical Fluid Dynamics

Rossby waves in an azimuthal wind

J. F. McKenzie[a] & G. M. Webb[b]

[a] School of Mathematics, Statistics and Physics, Durban University
of Technology, Durban, 4001, South Africa.

[b] Center for Space Plasma and Aeronomic Research, University of
Alabama in Huntsville, Huntsville, AL, 35805, USA.
Published online: 24 Nov 2014.

Click for updates

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis
Taylor & Francis Group

Rossby waves in an azimuthal wind

J. F. MCKENZIE† and G. M. WEBB*‡

†School of Mathematics, Statistics and Physics, Durban University of Technology, Durban 4001,
South Africa
‡Center for Space Plasma and Aeronomic Research, University of Alabama in Huntsville, Huntsville,
AL 35805, USA

Rossby waves in an azimuthal wind are analyzed using an eigen-function expansion. Solutions of the wave equation for the stream-function ψ for Rossby waves are obtained in which ψ depends on (r, ϕ, t) where r is the cylindrical radius, ϕ is the azimuthal angle measured in the β plane relative to the Easterly direction, (the β-plane is locally horizontal to the Earth's surface in which the x-axis points East, and the y-axis points North). The radial eigenfunctions in the β-plane are Bessel functions of order n and argument kr, where k is a characteristic wave number and have the form $a_n J_n(kr)$ in which the a_n satisfy recurrence relations involving a_{n+1}, a_n, and a_{n-1}. The recurrence relations for the a_n have solutions in terms of Bessel functions of order $n - \omega/\Omega$ where ω is the frequency of the wave and Ω is the angular velocity of the wind and argument $a = \beta/(k\Omega)$. By summing the Bessel function series, the complete solution for ψ reduces to a single Bessel function of the first kind of order ω/Ω. The argument of the Bessel function is a complicated expression depending on r, ϕ, a, and kr. These solutions of the Rossby wave equation can be interpreted as being due to wave-wave interactions in a locally rotating wind about the local vertical direction. The physical characteristics of the rotating wind Rossby waves are investigated in the long and short wavelength limits; in the limit as the azimuthal wind velocity $V_w \to 0$; and in the zero frequency limit $\omega \to 0$ in which one obtains a stationary spatial pattern for the waves. The vorticity structure of the waves are investigated. Time dependent solutions with $\omega \neq 0$ are also investigated.

Keywords: Rossby waves; Azimuthal wind; Fourier–Floquet

1. Introduction

The properties of Rossby waves on a rotating planet have been extensively elucidated in classical texts by Gill (1982), Pedlosky (1987) and Vallis (2006). The wave motion, which is of planetary scale arises from the latitudinal variation of the vertical component of the Coriolis force, known as the β-effect, and is closely related to the concept of the conservation of potential vorticity (see, e.g., Vallis 2006, pp. 178–183). The characteristic dispersion and anisotropic wave propagation features are probably best understood in terms of the wave normal curve introduced by Longuet-Higgins (1964). In wave number space $\boldsymbol{k} = (k_x, k_y)$, at a fixed frequency ω, this curve is a circle in $\boldsymbol{k}$-space, with center displaced westward ($k_x < 0$) along the k_x axis by an amount of $\beta/(2\omega)$, and with diameter β/ω. This implies that the phase

---

*Corresponding author. Email: gmw0002@uah.edu

propagation is entirely westward, However, in the case of a finite Rossby radius, it has recently been shown that the group velocity diagram is an ellipse which exhibits both eastward as well as westward energy propagation (Duba and McKenzie 2012). These features are also captured in the Green's function of the Rossby wave equation which can be expressed analytically as a cylindrical Hankel function describing an isotropic outgoing wave, on which is superimposed a westward phase propagating at the "Rossby" speed which corresponds to the center of the Longuet-Higgins circle in $\boldsymbol{k}$-space (Rhines 2003, McKenzie 2014).

The effects of winds has dramatic effects on the propagation properties through the Doppler shifting of the frequency which gives rise to changes in the topology of the $\boldsymbol{k}$ or wave normal diagram. One such important feature arising from the Doppler shift is the appearance of an asymptote (or blocking line) where the Doppler shift brings the frequency to zero and in the case of inhomogeneous media leads to the existence of a critical level (see, e.g., Dickinson 1968). An important property of the $\boldsymbol{k}$-diagram is that its normal gives the direction of energy propagation (Lighthill 1978). Using this property, together with the method of stationary phase, the radiation patterns of Rossby waves in both zonal and meridional winds have been constructed (Duba *et al.* 2014).

In this paper we analyze the effects of an azimuthal wind rotating with angular velocity $\Omega(r)$. The analysis is similar to that of Couette flow. The novel feature is that the Couette flow analysis (Chandrasekhar 1961) is conducted on a β-plane. Therefore, the usual Couette flow equation contains additional terms representing the asymmetric β effect. In cylindrical coordinates the classical Rossby wave equation takes the form of classical cylindrical waves in which the harmonics of the angular frequency, are coupled to neighboring harmonics through a three wave interaction process. This is demonstrated in the next section in which we transform the Rossby wave equation for the stream function $\psi(x, y, t)$ from its usual Cartesian form (x is east and y is north) to a cylindrical form (r, ϕ) in the presence of an azimuthal wind $\boldsymbol{U} = r\Omega(r)\boldsymbol{e}_\phi$ where r is the radius and ϕ is the azimuthal angle in the $x - y$ plane (see (5)). For the simple case in which $\Omega(r)$ is a constant, Fourier–Floquet analysis (section 3) shows that the radial structure functions $R_n(r)$ to this equation are Bessel functions of integer (harmonic) order n and argument kr, where k is the radial wave number, in which the amplitudes a_n of each $J_n(kr)$ are related through a three term recurrence relation between a_n, a_{n+1}, and a_{n-1}. The key coefficients in this scheme are the recurrence relation constants $c_n = (\beta/(2k))\omega_n^{-1}$ (see (10) and (15)), in which $\omega_n = \omega - n\Omega$ are the shifted frequencies. The dispersion relation between ω and k derived in section 3.1, for which we develop a continued fraction approximation scheme for the c_n, shows how the β-effect shifts the various harmonic frequencies due to the wind.

This solution is further developed in section 4, in which the Fourier analysis of the recursion relation shows that the harmonic amplitude a_n can be expressed in terms of Bessel functions $J_{n-\overline{\omega}}(a)$, where $a = \beta/(k\Omega)$ and $\overline{\omega} = \omega/\Omega$. On substituting these a_n in the Fourier–Floquet expansion it is found (remarkably) that the infinite series can be summed (Gradshteyn and Ryzhik 2000) to give a solution involving Bessel functions of order $\pm\overline{\omega}$ but with arguments D_1 (D_2) given by (43)–(48), which lends itself to a simple geometric interpretation. It is noteworthy that this solution can be directly verified by transforming the Rossby wave equation from the independent variables (r, ϕ, t) to new variables (D, p, t) which yields the wave equation (56), the solutions of which are indeed of the form deduced from summing the original infinite Fourier–Floquet series.

Section 5, investigates the physical characteristics of Rossby waves in an azimuthally rotating wind. We show that the solutions reduce to the planar westward, phase propagating Rossby waves expected in the absence of the wind. We also consider the limit of steady Rossby

waves in a rotating wind obtained as the wave frequency $\omega \to 0$, and investigate the vorticity structure of the wave. The short wave limit of the Rossby wave solutions (43) and (46) in an azimuthally rotating wind reveals a breather type solution at short wavelengths. At long wavelengths, the solutions reduce to the usual planar, westward phase propagating Rossby waves. We also study time dependent Rossby wave solutions with frequency $\omega \neq 0$.

Section 6 concludes with a summary and discussion.

2. Rossby wave equation in an azimuthal wind

The classic, linearized Rossby wave equation for the stream function $\psi(x, y, t)$ on a β-plane may be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla_h^2 \psi\right) + \beta \frac{\partial \psi}{\partial x} = 0, \tag{1}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla}, \qquad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}$$

$$f = 2\Omega_E \sin \theta_0 + \beta y, \qquad \beta = \frac{2\Omega_E}{R}\cos \theta_0. \tag{3}$$

Here the β-plane is centered at latitude θ_0 on a planet of radius R rotating with angular frequency Ω_E, and x and y are Cartesian coordinates pointing east and north, respectively.

The Rossby wave equation (1) is derived in Longuet-Higgins (1964), who also derives the radial vorticity conservation equation for planetary waves on a rotating sphere. He shows that the equation for the velocity potential $\Psi = \Psi(\Theta, \Phi, t)$ for waves on the rotating sphere [his equation (38)] has solutions in terms of spherical harmonics, where (R, Θ, Φ) are spherical polar coordinates in which the polar axis is along the rotation axis of the sphere (the analysis assumes $R = $ constant and $\boldsymbol{U} = \boldsymbol{0}$). This is a more exact formulation for Rossby waves, which does not assume the β-plane approximation. Longuet Higgins shows that the β-plane approximation gives a relatively good approximation to planetary waves on a rotating sphere [governed by his equation (38)], for high wave numbers n for the spherical harmonics, provided one is sufficiently far away from caustics. In the present paper, we use the approximate β-plane approach to Rossby waves on a rotating planet, in which there is an azimuthal wind at latitude $\theta = \theta_0$, which may be regarded as an approximation to the spherical harmonic approach of Longuet-Higgins (1964).

In this paper, we assume that the background wind velocity $\boldsymbol{U}$ is azimuthal and thus may be expressed in the form

$$\boldsymbol{U} = U_\phi \boldsymbol{e}_\phi = r\Omega(r)\boldsymbol{e}_\phi, \tag{4}$$

where $\boldsymbol{e}_\phi$ is the unit vector in the azimuthal direction. We use cylindrical coordinates in which

$$x = r\cos \phi, \qquad y = r\sin \phi. \tag{5}$$

Equation (1) then takes the form

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}\right] + \beta\left[\cos \phi \frac{\partial \psi}{\partial r} - \frac{\sin \phi}{r}\frac{\partial \psi}{\partial \phi}\right] = 0. \tag{6}$$

In the next section we seek separable Fourier type solutions of (6).

3. Fourier–Floquet solutions

In the simplest case where the angular velocity of the azimuthal wind Ω is independent of r, we seek solutions of (6) of the form

$$\psi(r, \phi, t) = \sum_{n=-\infty}^{\infty} R_n(r) \exp\big[\mathrm{i}(\omega t - n\phi)\big]. \tag{7}$$

Substituting the ansatz (7) into (6) and equating equal powers of $\exp(-\mathrm{i}n\phi)$ yields the following set of differential equations for the radial structure functions $R_n(r)$:

$$\mathrm{i}\omega_n\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_n}{\mathrm{d}r}\right) - \frac{n^2}{r^2}R_n\right]$$
$$+ \frac{\beta}{2}\left[\frac{\mathrm{d}R_{n+1}}{\mathrm{d}r} + \frac{\mathrm{d}R_{n-1}}{\mathrm{d}r} + \frac{1}{r}\big((n+1)R_{n+1} - (n-1)R_{n-1}\big)\right] = 0, \tag{8}$$

in which $\omega_n = \omega - n\Omega$ are the Doppler shifted frequencies. These coupled equations for R_n and $R_{n\pm1}$ suggest a solution for the R_n in the form of Bessel functions with

$$R_n = a_n \mathrm{J}_n(kr), \qquad\qquad R_{n\pm1} = a_{n\pm1}\mathrm{J}_{n\pm1}(kr), \tag{9}$$

in which the constants a_n and $a_{n\pm1}$ are connected through the recursion relation

$$\mathrm{i}\omega_n a_n = \frac{\beta}{2k}(a_{n+1} - a_{n-1}). \tag{10}$$

Here k is the radial wavenumber. One can verify that (9) and (10) do indeed satisfy (8). This follows by using the Bessel function recursion relations

$$\mathrm{J}'_{n\pm1}(\bar{r}) \pm \frac{n\pm1}{\bar{r}}\mathrm{J}_{n\pm1}(\bar{r}) = \pm\mathrm{J}_n(\bar{r}), \tag{11}$$

where $\bar{r} = kr$ in (11) (Abramowitz and Stegun 1965, formula (9.1.27), p. 361). The $\mathrm{J}_n(\bar{r})$ satisfy Bessel's equation

$$\frac{\mathrm{d}^2\mathrm{J}_n(\bar{r})}{\mathrm{d}\bar{r}^2} + \frac{1}{\bar{r}}\frac{\mathrm{d}\mathrm{J}_n(\bar{r})}{\mathrm{d}\bar{r}} + \left(1 - \frac{n^2}{\bar{r}^2}\right)\mathrm{J}_n(\bar{r}) = 0. \tag{12}$$

The recursion relations (10) may be written in the matrix form:

$$\mathsf{M}_{\mu\nu}a_\nu = \mathrm{i}c_{-N}a_{-N-1}\delta_{\mu,-N} - \mathrm{i}c_N a_{N+1}\delta_{\mu,N}, \qquad -N \le \mu, \quad \nu \le N, \tag{13}$$

where

$$\mathsf{M}_{\mu\nu} = \delta_{\mu,\nu} + \mathrm{i}c_\mu\big(\delta_{\nu,\mu+1} - \delta_{\nu,\mu-1}\big), \tag{14}$$

is an $(2N+1) \times (2N+1)$ tri-diagonal matrix, and

$$c_n = \frac{\beta}{2k\omega_n} = \frac{\Omega_E}{\Omega}\frac{\cos\theta_0}{kR}\frac{1}{(\bar{\omega}-n)}, \qquad n = 0, \pm1, \pm2, \ldots, \pm N. \tag{15}$$

As an approximation to close the system, we assume the source terms on the right-hand side of (13) for large enough N are negligible, in which case we obtain the approximate matrix equation

$$\mathsf{M}\mathsf{a} = \mathbf{0}, \qquad \text{where} \qquad \mathsf{a} = (a_{-N}, a_{1-N}, \ldots a_{N-1}, a_N)^T \tag{16}$$

is a $2N+1$ dimensional state vector describing the a_j.

The dispersion equation for the system (16) is given by

$$\det \mathsf{M} = 0, \tag{17}$$

which relates $\overline{\omega}$ to k for each harmonic n. In the next subsection, we discuss the dispersion equation in more detail. In the limit as $N \to \infty$ the dispersion equation (17) becomes exact.

3.1. *The dispersion relation*

The solutions given by (9) for the radial structure functions $R_n(kr)$, together with the recursion relations (10) for their amplitudes a_n satisfy the system (16) and the dispersion equation (17). For given values of $\Omega_E \cos \theta_0 / \Omega$ the dispersion relation (17) between $\overline{\omega}$ and the radial dimensionless wavenumber kr describes the interaction between the various harmonics $\omega = n\Omega$ through the three wave interactions expressed through the recursion relations for the system expressed by the tri-diagonal system (16). Below we look in more detail at the recursion relations (10) or equivalently, the matrix system (16).

The recursion relations (10) can be written in the form

$$\frac{a_n}{a_{n-1}} = \mathrm{i}c_n - \mathrm{i}c_n \frac{a_{n+1}}{a_{n-1}}, \tag{18}$$

$$\frac{a_n}{a_{n+1}} = -\mathrm{i}c_n + \mathrm{i}c_n \frac{a_{n-1}}{a_{n+1}}. \tag{19}$$

For $n = 0$ we obtain

$$1 = -\mathrm{i}c_0 \left(\frac{a_1}{a_0} - \frac{a_{-1}}{a_0} \right). \tag{20}$$

We use (18) to compute a_n/a_{n-1} for n a positive integer and (19) to compute a_n/a_{n+1} for n a negative integer. Thus, a_1/a_0 and a_{-1}/a_0 can be written in the form of continued fractions. To calculate a_1/a_0 and a_{-1}/a_0 as continued fractions, set

$$z_n = \frac{a_n}{a_{n-1}}, \qquad y_n = \frac{a_n}{a_{n+1}}, \tag{21}$$

then (18) and (19) can be written as

$$z_n \left(-\mathrm{i}/c_n + z_{n+1}\right) = 1, \qquad y_n \left(\mathrm{i}/c_n + y_{n-1}\right) = 1. \tag{22}$$

Using (22) we obtain

$$\frac{a_1}{a_0} = z_1 = \cfrac{1}{-\dfrac{\mathrm{i}}{c_1} + z_2} = \cdots = \cfrac{1}{-\dfrac{\mathrm{i}}{c_1} + \cfrac{1}{-\dfrac{\mathrm{i}}{c_2} + \cfrac{1}{-\dfrac{\mathrm{i}}{c_3} + \cdots}}}. \tag{23}$$

Similarly,

$$\frac{a_{-1}}{a_0} = y_{-1} = \cfrac{1}{\dfrac{\mathrm{i}}{c_{-1}} + y_{-2}} = \cdots = \cfrac{1}{\dfrac{\mathrm{i}}{c_{-1}} + \cfrac{1}{\dfrac{\mathrm{i}}{c_{-2}} + \cfrac{1}{\dfrac{\mathrm{i}}{c_{-3}} + \cdots}}}. \tag{24}$$

Note that for large n, $|c_{\pm n}| \propto 1/|(\overline{\omega} \mp n)| \to 0$ as $n \to \infty$. The basic theory for continued fractions is well documented (see, e.g., Jones 1980). The above continued fractions can be truncated to provide good approximations to a_1/a_0 and a_{-1}/a_0 which on substitution in (20) provides a relatively accurate approximation to the dispersion equation. Since each $c_n \propto 1/kR$, we expect relatively good convergence for $kR \gg 1$.

Consider, for example, the truncated three wave expansion for $n = 0, \pm 1$. From (23) and (24) we obtain

$$\frac{a_1}{a_0} \approx \frac{1}{-\mathrm{i}/c_1}, \qquad \frac{a_{-1}}{a_0} \approx \frac{1}{\mathrm{i}/c_{-1}}, \tag{25}$$

which, when substituted into (20), gives the approximate dispersion relation

$$1 = c_0(c_1 + c_{-1}) = \frac{2\alpha^2}{(\overline{\omega}^2 - 1)}, \qquad \text{where} \qquad \alpha = \frac{\Omega_E \cos\theta_0}{\Omega kR}, \tag{26}$$

and in turn gives

$$\overline{\omega}^2 = 1 + 2\alpha^2 \equiv 1 + 2\left(\frac{\Omega_E \cos\theta_0}{\Omega kR}\right)^2. \tag{27}$$

This latter dispersion relation demonstrates how the β effect shifts the $\overline{\omega} = \pm 1$ fundamental due to the wind in the limit $kR \gg 1$. The group and phase velocities of the wave are given by

$$\omega \approx \Omega\left(1 + \alpha^2\right), \qquad V_p = \frac{\omega}{k} = \frac{\Omega}{k}\left(1 + \alpha^2\right), \qquad V_g = \frac{\partial\omega}{\partial k} = -2\alpha^2\frac{\Omega}{k}. \tag{28}$$

This demonstrates that the group and phase velocities V_g and V_p are in opposite directions.

Below we look at the case, where the matrix M is a 5×5 matrix. We use the equivalent matrix system (10) to carry out the analysis, i.e. we consider the matrix system

$$\mathsf{P}\mathbf{a} = \mathbf{0}, \tag{29}$$

where P is the $(2N + 1) \times (2N + 1)$ system analogous to (16) in which we set $a_{N+1} = 0$ and $a_{-N-1} = 0$. In the 5×5 case we have

$$\mathsf{P} = \begin{pmatrix} -\mathrm{i}\omega_{-2} & b & 0 & 0 & 0 \\ -b & -\mathrm{i}\omega_{-1} & b & 0 & 0 \\ 0 & -b & -\mathrm{i}\omega_0 & b & 0 \\ 0 & 0 & -b & -\mathrm{i}\omega_1 & b \\ 0 & 0 & 0 & -b & -\mathrm{i}\omega_2 \end{pmatrix}, \tag{30}$$

where

$$b = \frac{\beta}{2k}. \tag{31}$$

Setting $\det \mathsf{P} = 0$, we obtain the dispersion equation

$$\begin{aligned} \det \mathsf{P} &= -\mathrm{i}\omega\left[\left(\omega^2 - 4\Omega^2\right)\left(\omega^2 - \Omega^2\right) - 4b^2\left(\omega^2 - \Omega^2\right) + 3b^4\right], \\ &\equiv -\mathrm{i}\Omega^5\overline{\omega}\left[(\overline{\omega}^2 - 1)(\overline{\omega}^2 - 4) - 4\alpha^2(\overline{\omega}^2 - 1) + 3\alpha^4\right] = 0. \end{aligned} \tag{32}$$

The dispersion relation (32) has solutions $\overline{\omega} = 0$ and four other solutions for $\overline{\omega}$. The two non-zero solutions for $\overline{\omega}^2$ can be obtained by using the quadratic formula to solve (32) for $\overline{\omega}^2$. Alternatively, using perturbation theory we obtain the solutions

$$\omega = \pm\Omega\left(1 + \frac{1}{2}\alpha^4 \cdots\right), \qquad \omega = \pm 2\Omega\left(1 + \frac{1}{2}\alpha^2 - \frac{5}{32}\alpha^4\right), \tag{33}$$

valid for $\alpha^2 \ll 1$ (i.e. for large kR). These solutions show the dependence of the fundamental and the second harmonic solutions on the parameter α.

Clearly, further examples of solutions of the truncated dispersion equation can be obtained for larger N by the same procedure. In the next section we show further aspects of the solution and find closed form solutions for the a_n coefficients in terms of Bessel functions. Then using a Neumann expansion for the products of Bessel functions leads to a closed form solution of (7) in terms of a single Bessel function, with a complicated argument which incorporates both the usual Rossby wave solutions, modified by the effects of the locally rotating wind.

4. Recurrence relations (10) and ψ

PROPOSITION 4.1 *The recurrence relations* (10) *have solutions for a_n of the form*

$$a_n = b_1 a_n^1 + b_2 a_n^2, \tag{34}$$

where b_1 and b_2 are arbitrary constants and a_n^j ($j = 1, 2$) are given by

$$a_n^1 = \exp\left[-\mathrm{i}\frac{\pi}{2}(n - \overline{\omega})\right] \mathrm{J}_{n-\overline{\omega}}(a), \qquad a_n^2 = \exp\left[-\mathrm{i}\frac{\pi}{2}(\overline{\omega} - n)\right] \mathrm{J}_{\overline{\omega}-n}(a) \tag{35}$$

with

$$\overline{\omega} = \frac{\omega}{\Omega} \qquad and \qquad a = \frac{\beta}{k\Omega}. \tag{36}$$

Proof Below we verify that a_n^1 satisfies the recurrence relations (10). One can also verify the solution for a_n^2 by the same methods. A detailed derivation of the formulae (35) for a_n^1 and a_n^2 using Fourier transforms is given in appendix A.

Using the Bessel function recurrence relations

$$\mathrm{J}_{\nu-1}(a) + \mathrm{J}_{\nu+1}(a) - \frac{2\nu}{a}\mathrm{J}_\nu(a) = 0 \tag{37}$$

(Abramowitz and Stegun 1965, the first formula in (9.1.27), p. 361) with $\nu = n - \overline{\omega}$, we obtain

$$\begin{aligned}
\frac{a}{2}&\left(a_{n+1}^1 - a_{n-1}^1\right) - \mathrm{i}\,(\overline{\omega} - n)\,a_n^1 \\
&= \frac{a}{2}\left\{\exp\left[-\mathrm{i}\frac{\pi}{2}\,(n+1-\overline{\omega})\right]\mathrm{J}_{n+1-\overline{\omega}}(a) - \exp\left[-\mathrm{i}\frac{\pi}{2}\,(n-1-\overline{\omega})\right]\mathrm{J}_{n-1-\overline{\omega}}(a)\right. \\
&\quad \left. - \frac{2\mathrm{i}(\overline{\omega}-n)}{a}\exp\left[-\mathrm{i}\frac{\pi}{2}\,(n-\overline{\omega})\right]\mathrm{J}_{n-\overline{\omega}}(a)\right\} \\
&\equiv \frac{a}{2}\exp\left[-\mathrm{i}\frac{\pi}{2}\,(n+1-\overline{\omega})\right]\left\{\mathrm{J}_{n+1-\overline{\omega}}(a) + \mathrm{J}_{n-1-\overline{\omega}}(a) - \frac{2(n-\overline{\omega})}{a}\mathrm{J}_{n-\overline{\omega}}(a)\right\} = 0,
\end{aligned} \tag{38}$$

where in the last step we used the Bessel recurrence relation (37). Equation (38) is equivalent to the re-currence relations (10) for the coefficients a_n. The derivation of (35) for the a_n^1 and a_n^2 using Fourier transforms is outlined in appendix A. □

From (34)–(37) the stream function $\psi(r, \phi, t)$ for Rossby waves in an azimuthal wind has solutions of the form

$$\psi(r, \phi, t) = \sum_{n=-\infty}^{\infty} a_n \mathrm{J}_n(kr) \exp\left[i(\omega t - n\phi)\right], \tag{39}$$

where the coefficients a_n have the general form (34). Using (34) for the a_n we obtain

$$\psi(r, \phi, t) = b_1\psi_1 + b_2\psi_2, \tag{40}$$

where b_1 and b_2 are arbitrary constants and

$$\psi_1 = \sum_{n=-\infty}^{\infty} \exp\left[-\mathrm{i}\frac{\pi}{2}(n - \overline{\omega})\right] \mathrm{J}_{n-\overline{\omega}}(a)\mathrm{J}_n(kr) \exp[i(\omega t - n\phi)], \tag{41}$$

$$\psi_2 = \sum_{n=-\infty}^{\infty} \exp\left[-\mathrm{i}\frac{\pi}{2}(\overline{\omega} - n)\right] \mathrm{J}_{\overline{\omega}-n}(a)\mathrm{J}_n(kr) \exp[i(\omega t - n\phi)], \tag{42}$$

where $\overline{\omega} = \omega/\Omega$.

PROPOSITION 4.2   *The series* (41) *for ψ_1 can be summed to obtain the solution form*

$$\psi_1 = \exp\left[\mathrm{i}\omega\left(t + \frac{\pi}{2\Omega}\right)\right] \exp(-\mathrm{i}\overline{\omega}p_1)\mathrm{J}_{-\overline{\omega}}(D_1), \tag{43}$$

where

$$\exp(2\mathrm{i}p_1) = \frac{a - kr \exp[\mathrm{i}(\pi/2 + \phi)]}{a - kr \exp[-\mathrm{i}(\pi/2 + \phi)]}, \tag{44}$$

$$D_1^2 = k^2r^2 + a^2 + 2kra\sin\phi. \tag{45}$$

Here $\mathrm{J}_{-\overline{\omega}}(D_1)$ *is a Bessel function of the first kind of order* $v = -\overline{\omega}$ *and argument* D_1.
    Similarly, the solution (42) *for $\psi_2(r, \phi, t)$ can be reduced to the form*

$$\psi_2 = \exp\left[\mathrm{i}\omega\left(t - \frac{\pi}{2\Omega}\right)\right] \exp(\mathrm{i}\overline{\omega}p_2)\mathrm{J}_{\overline{\omega}}(D_2), \tag{46}$$

where

$$\exp(2\mathrm{i}p_2) = \frac{a - kr \exp[\mathrm{i}(\phi - \pi/2)]}{a - kr \exp[-\mathrm{i}(\phi - \pi/2)]}, \tag{47}$$

$$D_2^2 = k^2r^2 + a^2 - 2kra\sin\phi, \tag{48}$$

and $\mathrm{J}_{\overline{\omega}}(D_2)$ *is a Bessel function of the first kind of order* $v = \overline{\omega}$.

Proof The formulae (43)–(48) follow from formulas (8.5.30), p. 930 of Gradshteyn and Ryzhik (2000), who give a summation formula for Bessel functions with a geometric interpretation. They consider a triangle ABC. They use the notation $r = AC$, $\rho = AB$, $D = BC$ where it is assumed that $\rho < r$. They also use the notation $\varphi = A$ and $\psi = C$ to denote the angles of the triangle at the vertices A and C, and assume that $0 < \psi < \pi/2$ (i.e. $0 < C < \pi/2$). By the cosine rule, $D = \sqrt{r^2 + \rho^2 - 2r\rho\cos\varphi}$. They also give the formula

$$\exp(2\mathrm{i}\psi) = \frac{r - \rho \exp(-\mathrm{i}\varphi)}{r - \rho \exp(\mathrm{i}\varphi)}, \tag{49}$$

The angle ψ can also be obtained by using the sine rule or the cosine rule for the triangle ABC, i.e.

$$\frac{\sin\psi}{\rho} = \frac{\sin\varphi}{D} \qquad \text{and} \qquad \cos\psi = \frac{D^2 + r^2 - \rho^2}{2Dr}. \tag{50}$$

When the above conditions are satisfied one obtains the summation formula

$$\exp(\mathrm{i}v\psi)Z_v(mD) = \sum_{k=-\infty}^{\infty} \mathrm{J}_k(m\rho)Z_{v+k}(mr) \exp(\mathrm{i}k\varphi), \tag{51}$$

where m is an arbitrary complex number and $Z_\nu(z)$ is one of the solutions of Bessel's equation (e.g. $Z_\nu = J_\nu$, $Z_\nu = Y_\nu$, $Z_\nu = H_\nu^1$ or $Z_\nu = H_\nu^2$, where J_ν is a Bessel function of the first kind, Y_ν is a Bessel function of the second kind, and H_ν^p ($p = 1, 2$) are Hankel functions. In the case $Z_\nu = J_\nu$ the restriction $\rho < r$ is superfluous.

Use of formulae (49)–(51) with appropriate values of the parameters, gives the solutions (43) and (46) for ψ_1 and ψ_2. We use the notation $p_j = \psi_j$ ($j = 1, 2$) to denote the angle ψ in our applications. In the derivation of (43) we set $\varphi = -\pi/2 - \phi$ and $\nu = -\overline{\omega}$ and $Z_\nu = J_\nu$. To derive (46) we set $\varphi = \pi/2 - \phi$ and $\nu = \overline{\omega}$. $\qquad\square$

The verification of the solutions (43) and (46) can be obtained by transforming (6) for the Rossby wave stream function ψ from the independent variables (r, ϕ, t) to the independent variables (D, p, t) (i.e. $p = p_1$ and $D = D_1$ for the solution (43) and $\varphi = -\pi/2 - \phi$). We find

$$\nabla_\perp^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = k^2\left(\frac{\partial^2 \psi}{\partial D^2} + \frac{1}{D}\frac{\partial \psi}{\partial D} + \frac{1}{D^2}\frac{\partial^2 \psi}{\partial p^2}\right) \equiv k^2 \widetilde{\nabla}_\perp^2 \psi, \tag{52}$$

for the 2D Laplacian. Thus the 2D Laplacian preserves its basic structure under the transformations, except for a scaling factor of k^2. Similarly, we obtain

$$\beta\left(\cos\phi\frac{\partial \psi}{\partial r} - \frac{\sin\phi}{r}\frac{\partial \psi}{\partial \phi}\right) = \frac{\beta k}{D^2}\left[(kr\cos\varphi - a)\frac{\partial \psi}{\partial p} - krD\frac{\partial \psi}{\partial D}\right]$$
$$= -\beta k\left(\sin p\frac{\partial \psi}{\partial D} + \frac{\cos p}{D}\frac{\partial \psi}{\partial p}\right), \tag{53}$$

and

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\right) \equiv \frac{\partial}{\partial t} + \Omega\left[-a\sin p\frac{\partial}{\partial D} + \left(1 - \frac{a}{D}\cos p\right)\frac{\partial}{\partial p}\right], \tag{54}$$

where $a = \beta/(k\Omega)$. Using (52)–(54) the Rossby wave equation (6) reduces to

$$k^2\left(\frac{\partial}{\partial t} + \Omega\left[-a\sin p\frac{\partial}{\partial D} + \left(1 - \frac{a}{D}\cos p\right)\frac{\partial}{\partial p}\right]\right)\left(\frac{\partial^2 \psi}{\partial D^2} + \frac{1}{D}\frac{\partial \psi}{\partial D} + \frac{1}{D^2}\frac{\partial^2 \psi}{\partial p^2}\right)$$
$$- \beta k\left(\sin p\frac{\partial \psi}{\partial D} + \frac{\cos p}{D}\frac{\partial \psi}{\partial p}\right) = 0. \tag{55}$$

Equation (55) can also be written as

$$k^2\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial p}\right)\widetilde{\nabla}_\perp^2 \psi - \beta k\left(\sin p\frac{\partial}{\partial D} + \frac{\cos p}{D}\frac{\partial}{\partial p}\right)\left(\widetilde{\nabla}_\perp^2 \psi + \psi\right) = 0. \tag{56}$$

Equation (56) admits solutions of the form (43), i.e. of the form $\psi = \exp[\mathrm{i}(\omega t - \overline{\omega}p)]J(D)$, where $J(D)$ satisfies Bessel's equation of order $\overline{\omega}$. Note if $\overline{\omega}$ is an integer, i.e. $\omega = m\Omega$ then the solution of Bessel's equation is of the form $J = AJ_m(D) + BY_m(D)$, where A and B are arbitrary constants and $Y_m(D)$ is a Bessel function of the second kind (for m an integer, $Y_m(D)$ possesses a logarithmic term in its expansion (Abramowitz and Stegun 1965, Chap. 9, p. 359). A similar analysis applies to the solution (46). The present analysis, based on the solutions of (56), suggest that the solutions for the rotating wind Rossby wave equation can also contain a second independent solution of Bessel's equation, which is not obvious from the analysis based on the recurrence formulae solutions (35) for the a_n.

5. Solution characteristics and applications

In this section, we investigate the physical characteristics of the solutions (43) and (46) for ψ_1 and ψ_2 in Proposition 4.2. For the rotating wind case, the wind velocity is of the form: $\boldsymbol{u} = r\Omega\boldsymbol{e}_\phi$, where Ω is the angular velocity of the wind (see (4)).

5.1. *Zero rotation limit and steady-state limit*

PROPOSITION 5.1   *In the limit $\Omega \to 0$, the solution* (43) *for the rotating wind Rossby wave reduces to the plane wave solution*

$$a_1\psi_1 = a_1 A_1 \exp\big[i(\omega t - k_x x)\big], \tag{57}$$

where

$$A_1 \sim \exp\left(i\frac{\overline{\omega}\pi}{2}\right) J_{-\overline{\omega}}\left(\frac{\beta}{k\Omega}\right), \tag{58}$$

in the limit as $\overline{\omega} \to \infty$ and $\beta/(k\Omega) \to \infty$ (i.e. as $\Omega \to 0$). Note that the Bessel function argument $\beta/(k\Omega) \to \infty$ and the Bessel function order $-\overline{\omega} \to \infty$ in this limit (note for Rossby waves $\omega = -\beta k_x/k^2 < 0$). Using the asymptotic formulas for Bessel functions of large order and large argument (e.g., Abramowitz and Stegun 1965, *formula 9.3.3, p. 366), we obtain*

$$A_1 \sim \exp\left(i\frac{\overline{\omega}\pi}{2}\right)\sqrt{\frac{2}{\pi|\overline{\omega}|\tan\chi}}\cos\left[|\overline{\omega}|(\tan\chi - \chi) - \frac{\pi}{4}\right], \tag{59}$$

where $|\overline{\omega}| = |\omega/\Omega| \gg 1$ and

$$\cos\chi = \frac{k_x}{k}. \tag{60}$$

We can choose a_1 such that $a_1 A_1 = 1$, in which case (57) *reduces to a unit amplitude plane Rossby wave travelling to the west. The Rossby wave dispersion equation in the present case has the form*

$$\omega = -\frac{\beta k_x}{k^2}. \tag{61}$$

In the limit $\Omega \to 0$ and $\overline{\omega} = \omega/\Omega \to \infty$, (46) *reduces to the approximate solution*

$$a_2\psi_2 \sim a_2 \exp\left(-i\frac{\overline{\omega}\pi}{2}\right) J_{\overline{\omega}}\big(\alpha|\overline{\omega}|\big)\exp\big[i(\omega t - k_x x)\big], \tag{62}$$

where

$$\alpha = \frac{k}{k_x} > 1. \tag{63}$$

The amplitude a_2 in (62) *can be chosen so that* (62) *reduces to a planar Rossby wave of unit amplitude travelling westward, where the dispersion equation* (61) *describes the wave. Note that $\overline{\omega} = -|\overline{\omega}| \to -\infty$ as $\Omega \to 0$ in* (62).

PROPOSITION 5.2   *In the limit as $\omega \to 0$, the solutions* (43) *and* (46) *reduce to the steady state Rossby wave solutions in an azimuthal wind of the form:*

$$\psi_1 = J_0(D_1) \qquad and \qquad \psi_2 = J_0(D_2), \tag{64}$$

where D_1 and D_2 are given by (45) *and* (48) *respectively.*

5.2. *Short and long wavelength limits*

In the short wavelength limit, $(kr \gg 1)$, the solution for ψ_1 reduces to the formula

$$\psi_1 \sim \exp[i\omega(t - \phi/\Omega)]\mathrm{J}_{-\overline{\omega}}(kr). \tag{65}$$

By using the asymptotic formula

$$\mathrm{J}_{-\overline{\omega}}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos\left(kr + \frac{\overline{\omega}\pi}{2} - \frac{\pi}{4}\right) \tag{66}$$

([Abramowitz and Stegun 1965](#), formula 9.2.1, p. 364), the approximate real solution for ψ_1 from (65) and (66) is

$$\psi_1 \sim \sqrt{\frac{2}{\pi kr}} \cos\left[\omega\left(t - \frac{\phi}{\Omega}\right)\right] \cos\left(kr + \frac{\overline{\omega}\pi}{2} - \frac{\pi}{4}\right). \tag{67}$$

The solution (67) is a standing wave, with a fixed spatial envelope, which oscillates in time (i.e. (67) is a breather solution). The solution (67) can be decomposed into the sum of an inward and outward propagating wave, by using the trignometric formula

$$\cos A \cos B = \tfrac{1}{2}\big[\cos(A + B) + \cos(A - B)\big], \tag{68}$$

i.e.

$$\psi_1 \sim \frac{1}{\sqrt{2\pi kr}}\Big\{\cos\Big[\omega\Big(t - \frac{\phi}{\Omega}\Big) + kr + \frac{\pi\omega}{2\Omega} - \frac{\pi}{4}\Big] + \cos\Big[\omega\Big(t - \frac{\phi}{\Omega}\Big) - kr - \frac{\pi\omega}{2\Omega} + \frac{\pi}{4}\Big]\Big\}. \tag{69}$$

The balance between the outward and ingoing waves gives the breather solution (67).

Similarly, the solution (46) for $kr \gg 1$ gives the approximate real solution for ψ_2 as

$$\psi_2 \sim \sqrt{\frac{2}{\pi kr}} \cos\Big[\omega\Big(t + \frac{\phi}{\Omega}\Big) - \frac{\pi\omega}{\Omega}\Big]\cos\Big(kr - \frac{\overline{\omega}\pi}{2} - \frac{\pi}{4}\Big), \tag{70}$$

which is a standing breather type wave, but with a different dependence on ϕ than the solution (67).

The solution (43) for $a_1\psi_1$ in the long wavelength limit $(k \to 0)$, reduces to the planar, westward propagating Rossby wave (57)–(59). Similarly, the solution (46) for $a_2\psi_2$ in the long wavelength limit $(k \to 0)$ reduces to the planar, westward propagating Rossby wave (62).

5.3. *Typical parameter values*

Typical values of the parameters in the model for Rossby waves on Earth are: $\Omega_E \sim 2\pi/(24 \times 3600) \sim 7.27 \times 10^{-5}$ sec^{-1} for the angular speed of rotation of the Earth. The Rossby number $Ro =$ inertial acceleration/Coriolis acceleration $\sim (U^2/L)/(2\Omega U) \sim U/(2\Omega L)$. For a typical pressure field in the troposphere, with $L \sim 1000$ km and $U \sim 20$ m s^{-1}, the Rossby number $Ro \sim 0.1375$ (e.g., [Pedlosky 1987](#), p. 3). The β for Rossby waves from (9) is defined as $\beta = 2\Omega_E \cos\theta_0/R_E$. Taking a mean radius for the Earth of $\bar{R}_E \sim 6371$ km we obtain $\beta \sim 4.3 \times 10^{-12}$ m^{-1} s^{-1} at the equator and $\beta \sim 2.13 \times 10^{-12}$ m^{-1} s^{-1} at $\theta_0 = 60°$ latitude. For $L \sim 1000$ km structure, the wave number is $k_L \approx 6.28 \times 10^{-6}$ m^{-1}. For $L \sim 1000$ km, and a wind speed $V_W = 80$ km s^{-1}, at $r = L$, we obtain $\Omega = V_w/L = 0.08$ for the parameter Ω (note that $\Omega \gg \Omega_E$ by about a factor of 1000 in this case).
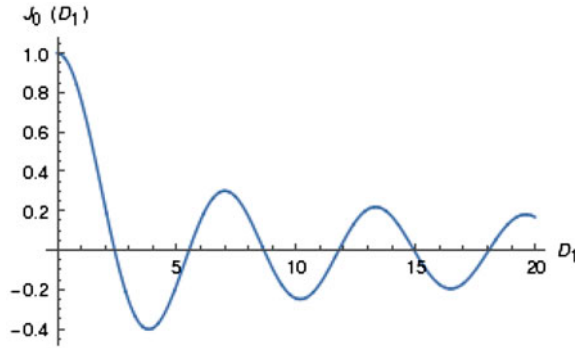
Figure 1. The stream function $\psi_1 = J_0(D_1)$ vs. D_1 for the steady wind solution (64) The z-vorticity $\omega = k^2\psi_1(D_1)$ is proportional to ψ_1. Note the vorticity changes sign at the zeros of the Bessel function $J_0(D_1)$ (Color online).

Consider the steady state Rossby wave solution in an azimuthal wind in (64). This solution depends on the spatial variable D_1 given in (45), which can be expressed in the form

$$D_1 = k\left[\left(y + \frac{\beta}{k^2\Omega}\right)^2 + x^2\right]^{1/2}. \tag{71}$$

The streamlines for the solution are given by $\psi_1 = J_0(D_1) = $ constant, which implies $D_1 = $ constant on the streamlines. Thus, the streamlines consist of circles of radius D_1 centered on the point $(x, y) = (0, -\beta/(k^2\Omega))$. The flow velocity $\boldsymbol{u}$ for the wave is given by

$$\boldsymbol{u} = \nabla\psi \times \hat{\boldsymbol{z}} = (\psi_{,y}, -\psi_{,x}, 0) = -\frac{k^2 J_1(D_1)}{D_1}\left(y + \frac{\beta}{k^2\Omega}, -x, 0\right), \tag{72}$$

which is tangent to the contours $\psi = \psi_1 = $ constant.

The local fluid vorticity in this model is given by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = -\nabla_\perp^2\psi\,\hat{\boldsymbol{z}}, \tag{73}$$

and the Rossby wave equation (1) is equivalent to the linearized vorticity equation

$$\frac{\partial\boldsymbol{\omega}'}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}') + \beta u_y'\hat{\boldsymbol{z}} \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)\boldsymbol{\omega}' + \beta u_y'\hat{\boldsymbol{z}} = \boldsymbol{0}. \tag{74}$$

Here we use the superscript $'$ to denote linearized wave quantities. Using the result (52) for $\nabla_\perp^2\psi$ and noting $\psi_1 = J_0(D_1)$ and $\partial\psi_1/\partial p = 0$, (73) gives

$$\boldsymbol{\omega} = k^2\psi_1(D_1)\hat{\boldsymbol{z}} = k^2 J_0(D_1)\hat{\boldsymbol{z}} \tag{75}$$

for the linearized wave vorticity. From (75) the fluid vorticity changes sign at the zeros of the J_0 Bessel function, i.e. at the points where $D_1 = j_{0n}$, where j_{0n} denotes the nth Bessel function zero where $J_0(j_{0n}) = 0$. From (71) this occurs when (x, y) lies on the circles

$$j_{0n}^2 = k^2\left[\left(y + \frac{\beta}{k^2\Omega}\right)^2 + x^2\right], \qquad n = 1, 2, \dots. \tag{76}$$

Figure 1 shows a plot of the stream function $\psi_1(D_1) = J_0(D_1)$ vs. D_1. This is essentially a rescaled version of the fluid vorticity (75). The fluid vorticity changes sign when D_1 is a zero of the J_0 Bessel function.
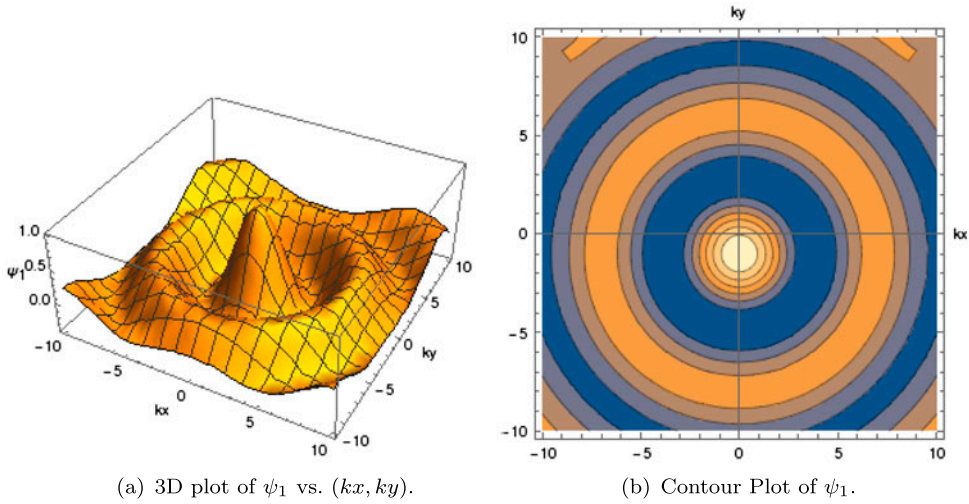
(a) 3D plot of ψ_1 vs. (kx, ky).

(b) Contour Plot of ψ_1.

Figure 2. Stream-function ψ_1 vs. (kx, ky) for the steady-state Rossby wave solution (64). (a) shows a 3D plot of ψ_1 vs. (kx, ky), and (b) shows the contour plots of ψ_1 in the $kx - ky$ plane. The vorticity is positive in the light orange-shaded regions and negative in the blue dark-shaded regions (Color online).

Figure 2(a) shows the vorticity distribution (75) as a 3D plot of ψ_1 vs. (kx, ky). For the sake of illustration, we consider the case

$$\psi_1 = J_0(D_1), \qquad D_1 = \left[(kx)^2 + (ky + a)^2\right]^{1/2}, \qquad a = \frac{\beta}{k\Omega}, \qquad (77)$$

where we set $a = 1$. Figure 2(b) shows the same stream function ψ_1 as a contour plot. The light shaded contours correspond to regions of positive vorticity, and the dark shaded regions to negative vorticity. Notice that the contours are circles centered on the point $(kx, ky) = (0, -1)$ in the $kx - ky$ plane. For $a = \beta/(k\Omega) = 1$ we obtain $\Omega = \beta/k$. Taking $\beta = 2.13 \times 10^{-12}$ m^{-1}s^{-1} for $\theta_0 = 60°$, and for $k = 6.28 \times 10^{-6}$ m^{-1}, we obtain $\Omega = 3.392 \times 10^{-7}$ s^{-1}, which corresponds to an extremely small wind velocity $V_w = \Omega L \sim 0.33$ms^{-1}. Thus, the negative displacement of the center of the contours down the ky-axis in figure 2(b) is only substantial for very small wind velocities. As $\Omega \to 0$, $a \to \infty$ and the center of the circles recedes to $(x, y) = (0, -\infty)$, and the curvature of the circles goes to zero (i.e. we obtain a plane Rossby wave in this limit).

In the more general, time dependent cases in (43) and (46), the vorticity of the fluid is also given by

$$\boldsymbol{\omega} = -\nabla_\perp^2 \psi \, \hat{z} = k^2 \psi \, \hat{z}. \qquad (78)$$

Thus, for example, for the solution (43) we obtain

$$\boldsymbol{\omega} = k^2 \psi_1 \hat{z}, \qquad \text{where} \qquad \psi_1 = \cos\left[\omega\left(t - \frac{p_1}{\Omega}\right) + \frac{\pi\overline{\omega}}{2}\right] J_{-\overline{\omega}}(D_1),$$

$$(79)$$

and we have taken the real part of the solution (43) to describe the wave. Thus, in the general time dependent case, a snapshot of the vorticity of the wave at a fixed time $t = t_1$ say, is more complicated because of the dependence of the solution on p_1. However, the steady state vorticity pattern is obtained on the manifold $t - p_1/\Omega = $ constant.
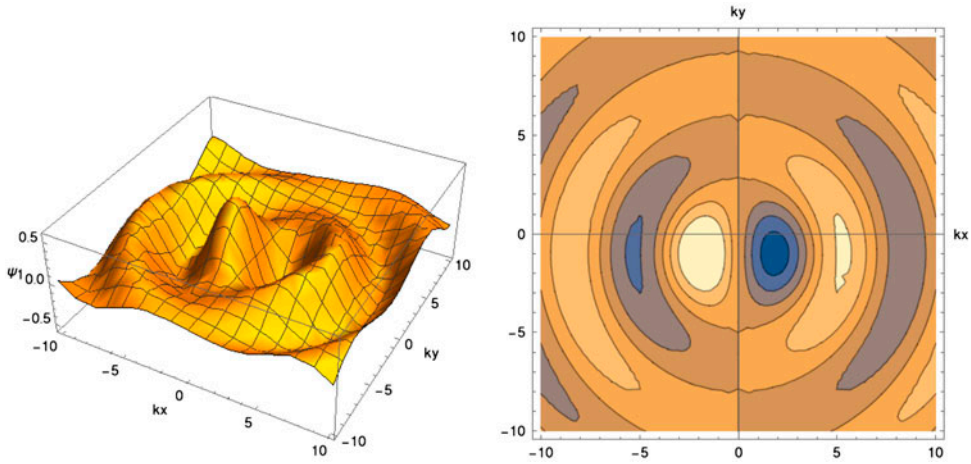
(a) 3D plot of ψ_1 vs. (kx, ky) at $t = 0$ for (83)   (b) Contour Plot of ψ_1 at $t = 0$ for (83)

Figure 3.   Stream-function ψ_1 vs. (kx, ky) for the Rossby wave solution (83), at time $\Omega t = 0$. The parameter $a = 1$. (a) shows a 3D plot of ψ_1 vs. (kx, ky), and (b) shows the contour plots of ψ_1 in the $kx - ky$ plane. The vorticity is positive in the light orange shaded regions and negative in the dark blue-shaded regions (Color online).

To evaluate ψ_1 in (79), we note from (44) that

$$\exp(2ip_1) = \frac{T}{T^*} = \frac{T^2}{|T|^2}, \tag{80}$$

where

$$T = a - kr \exp(-i\varphi), \qquad \varphi = -\frac{\pi}{2} - \phi. \tag{81}$$

Taking the positive square root of (80), i.e. $\exp(ip_1) = T/|T|$, we obtain

$$\cos p_1 = \frac{a + ky}{D_1}, \qquad \sin p_1 = -\frac{kx}{D_1}, \tag{82}$$

where D_1 is given by (77). Using (82) allows us to determine p_1 as a function of x and y. In particular for the case $\omega = -\Omega$, ψ_1 in (79) reduces to

$$\psi_1 = -\left[(ky + a)\sin(\Omega t) + kx\cos(\Omega t)\right]\frac{J_1(D_1)}{D_1}. \tag{83}$$

Figure 3(a) shows the vorticity distribution for the time dependent solution (83) for ψ_1 at time $\Omega t = 0$. The parameter $a = \beta/(k\Omega) = 1$. The corresponding contour plot of ψ_1 is shown in figure 3(b). Unlike the steady solution in figures 2(a),(b), the stream function ψ_1 and vorticity ($\propto \psi_1$) are now asymmetric with respect to the kx and ky axis. Figures 4(a),(b) show the same solution, but at the later time $\Omega t = \pi/4$. The vorticity pattern has clearly rotated by a phase angle of $\pi/4$. Clearly other complex vorticity patterns can be obtained from the more general solution (79).
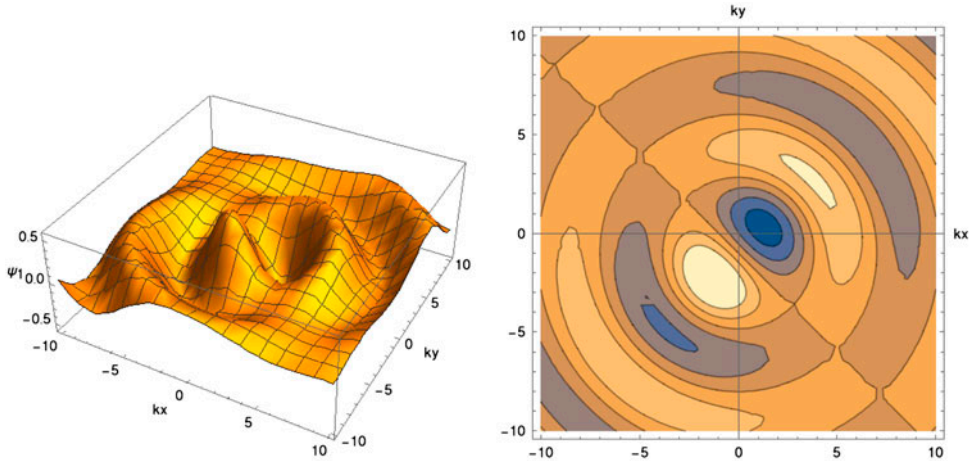
(a) 3D plot: ψ_1 vs. (kx, ky), $\Omega t = \pi/4$ for (83)   (b) Contour Plot of ψ_1 at $\Omega t = \pi/4$ for (83)

Figure 4.   Stream-function ψ_1 vs. (kx, ky) for the Rossby wave solution (83), at time $\Omega t = \pi/4$. The parameter $a = 1$. (a) shows a 3D plot of ψ_1 vs. (kx, ky), and (b) shows the contour plots of ψ_1 in the $kx - ky$ plane. The vorticity is positive in the light orange shaded regions and negative in the blue dark-shaded regions (Color online).

6. Conclusions

In this paper, we investigated a model for Rossby waves on a rotating planet, in which the waves are also subject to a rotating wind in the local β-plane. As in the usual linear Rossby wave model, the fluid vorticity can be split up into a local vorticity component, which interacts with the planetary or global vorticity, i.e. the rotation parameter is split up into a global vorticity component, dependent on the rotation rate of the planet, plus a local vorticity component in the β-plane. The β-plane consists of the tangent plane to the surface of the planet with normal perpendicular to the surface. The β-plane at latitude $\theta = \theta_0$ is locally described by the tangent plane in which the x-axis points east, the y-axis points north, and the local z-axis is perpendicular to the surface. In the present paper, the effect of an azimuthal wind in the β-plane, rotating from east to north is also included in the Rossby wave equation for the stream function. By searching for solutions for the stream function $\psi = \psi(r, \phi, t)$ where $x = r \cos \phi$ and $y = r \sin \phi$ that are separable in r and ϕ leads to solutions as a sum of Bessel functions for the nth harmonic of the form $\psi = \sum_{n=-\infty}^{\infty} R_n(r) \exp[i(\omega t - n\phi)]$. It turns out that solutions for the Rossby wave in a locally rotating wind exist where the radial eigen-functions have the form $R_n(r) = a_n J_n(kr)$ and the coefficients a_n satisfy coupled three term recurrence relations. We discuss how the coupling of the waves of different orders in n are related by tri-diagonal, truncated matrix systems of order N. A continued fraction solution method for the a_n was developed. However, it was also noticed, that the recurrence relations for the a_n could also be solved by a Fourier transform technique, yielding solutions for the a_n in terms of Bessel functions. By using a summation formula for the Bessel series for the complete solution given in (Gradshteyn and Ryzhik 2000, p. 930, formula (8.5.30)), one obtains closed form solutions involving a linear combination of Bessel functions of order $\pm \overline{\omega}$ where $\overline{\omega} = \omega/\Omega$ where Ω is the rotation frequency of the rotating wind. A direct verification of the solution was demonstrated by converting the Rossby wave operator and the 2-D Laplacian $\nabla_{\perp}^2 \psi$ by transformations from

the cylindrical polar coordinates in the (r, ϕ) in the β plane to new coordinates $D = D(\phi, r)$ and $p = p(\phi, r)$, in which the Laplacian is rescaled by a factor of k^2 in the new variables. This suggests that the new solution in the (D, p, t) coordinates is related to the symmetry group of the stream function equation for ψ. The exact nature of these symmetry transformations, is unclear, and remains to be investigated as an interesting problem in its own right.

The physical characteristics of the azimuthal rotating wind, Rossby wave solutions (43) and (46) were investigated in section 5. In the limit as the wind velocity $V_w \to 0$ (i.e. $\Omega \to 0$), the solutions reduce to planar Rossby waves with westward travelling phase velocity. The group velocity of Rossby waves is distinct from the phase velocity and has been described recently in the works by Duba *et al.* (2014) and McKenzie (2014). The dispersion equation for the waves in this limit is the usual Rossby wave dispersion equation (61). Both rotating wind Rossby wave solutions (43) and (46) reduce to westward propagating Rossby waves in the limit as $V_w \to 0$ and $\Omega \to 0$.

In the limit as $\omega \to 0$, the rotating wind Rossby waves (43) and (46) reduce to the steady Rossby wave solutions (64). These solutions show the combined effect of the rotating wind via the angular velocity Ω of the wind, and the Rossby wave parameter, β, where $\beta = 2\Omega_E \cos\theta_0/R_E$ describes the effect of the Coriolis force on waves with wavenumber k. The solutions show characteristic, annular circular bands in the $x - y$ plane of alternating positive and negative vorticity $\boldsymbol{\omega} = -\nabla_\perp^2 \psi \hat{z} = k^2 \psi \hat{z}$ depending on the value of D_1. For the solution (43) the circular annuli are centered on the point $(x, y) = (0, -\beta/(k^2\Omega))$ in the β-plane, with radii $r_n = j_{0n}/k$ (for the solution (46) the annuli are centered on $(0, \beta/(k^2\Omega))$).

In the short wavelength limit, $(k \gg 1)$, the solutions (43) and (46) reduce to breather type solutions, which can be thought of as the superposition of ingoing and outgoing waves that oscillate in both space and time. In the long wavelength limit, the solutions (43) and (46) reduce to the standard, planar westward propagating Rossby waves described in (57) et seq.

Figures 3 and 4 show the vorticity pattern for the time dependent Rossby wave solution (83) at times $\Omega t = 0$ and $\Omega t = \pi/4$. These vorticity patterns clearly rotate with increasing time Ωt and have an asymmetric vorticity pattern in the $kx - ky$ plane.

Acknowledgements

References

Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, 1965 (Dover: New York).

Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, 1961 (Oxford University Press: London).

Dickinson, R.E., Planetary Rossby waves propagating vertically through weak westerly wind wave guides. *J. Atmos. Sci.* 1968, **25**, 984–1002.

Duba, C.T., Doyle, T.B. and McKenzie, J.F., Rossby wave patterns in zonal and meridional winds. *Geophys. Astrophys. Fluid Dyn.* 2014, **108**, 237–257. doi:10.1080/03091929.2013.867604.

Duba, C.T. and McKenzie, J.F., Properties of Rossby waves for latitudinal β-plane variations of f and zonal variations of the shallow water speed. *Ann. Geophys.* 2012, **30**, 849–855. doi:10.5194/angeo-30-849-2012.

Gill, A.E., *Atmosphere and Ocean Dynamics*, 1982 (Academic Press: London).

Gradshteyn, I.S. and Ryzhik, I.M., Tables of integrals series and products 6th Edition, edited by A. Jeffrey, D. Zwillinger, 2000 (Academic Press: San Diego, London. ISBN 0-12-294757-6).

Jones, W.B. and Thron, W.J., Continued fractions: analytical theory and applications. In *Encyclopedia of Mathematics and Its Applications*. Vol. 11, 1980 (Addison Wesley Publishing Co.: Reading, MA. ISBN 0201-13510-8).

Lighthill, M.J., *Waves in Fluids*, 1978 (Cambridge University Press: Cambridge).

Longuet-Higgins, M.S., 1964, Planetary waves on a rotating sphere. *Proc. Roy. Soc. Lond. A* 1964, **279**, 446–473.

McKenzie, J.F., Group velocity and radiation pattern of Rossby waves. *Geophys. Astrophys. Fluid Dyn.* 2014, **108**, 225–268. doi:10.1080/03091929.2014.896459.

Pedlosky, J., *Geophysical Fluid Dynamics*, 1978 (Springer Verlag: New York).

Rhines, P.B., Rossby waves. In *Encylopedia of Atmospheric Sciences*, edited by J.R. Holton, J.A. Curry, J.A. Pyle, pp. 1–37, 2003 (Academic Press: Oxford).

Vallis, G., *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation*, 2006 (Cambridge University Press: Cambridge).

Appendix A

In this appendix we derive the solutions (35) of the recurrence relations (10) for the a_n^1 using Fourier transforms. The solutions for a_n^2 follow in a similar fashion by using a similar transform method in which the Fourier transform variable λ is replaced by $-\lambda$.

To derive (35) for a_n^1, we first re-write (10) in the form

$$a_n = a_{n-2} + [\zeta + \mu(n-1)]a_{n-1}, \tag{A.1}$$

where

$$\mu = -\frac{i\Omega}{\bar{\beta}}, \qquad \zeta = \frac{i\omega}{\bar{\beta}}, \qquad \bar{\beta} = \frac{\beta}{2k}. \tag{A.2}$$

To obtain solutions of (A.1), we think of n as a continuous variable t, and re-write (A.1) in the form

$$a(t) = a(t-2) + [\zeta + \mu(t-1)]a(t-1). \tag{A.3}$$

We introduce the Fourier transform

$$\bar{a}(\lambda) = \int_{-\infty}^{\infty} \exp(-i\lambda t)a(t)\, dt \equiv \mathcal{F}_\lambda[a(t)]. \tag{A.4}$$

By noting that

$$\mathcal{F}_\lambda[a(t-2)] = \exp(-2i\lambda)\bar{a}(\lambda), \qquad \mathcal{F}_\lambda[a(t-1)] = \exp(-i\lambda)\bar{a}(\lambda),$$

$$\mathcal{F}_\lambda[(t-1)a(t-1)] = \exp(-i\lambda)\mathcal{F}_\lambda[ta(t)] = i\exp(-i\lambda)\frac{d\bar{a}(\lambda)}{d\lambda}, \tag{A.5}$$

the difference equation for $a(t)$ in transform space becomes

$$\bar{a}(\lambda) = \exp(-2i\lambda)\bar{a}(\lambda) + \zeta\exp(-i\lambda)\bar{a}(\lambda) + i\mu\exp(-i\lambda)\frac{d\bar{a}(\lambda)}{d\lambda}. \tag{A.6}$$

Equation (A.6) may be written in the form

$$\frac{d\bar{a}}{d\lambda} + \frac{\bar{a}}{\mu}\left(-i\zeta - 2\sin\lambda\right) = 0. \tag{A.7}$$

The integrating factor I for (A.7) is

$$I = \exp\left(\int \frac{(-i\zeta - 2\sin\lambda)}{\mu}\, d\lambda\right) \equiv \exp\left(\frac{-i\zeta\lambda + 2\cos\lambda}{\mu}\right). \tag{A.8}$$

Thus, the solution for $\bar{a}(\lambda)$ has the form

$$\bar{a}(\lambda) = c_1\exp\left(\frac{i\zeta\lambda - 2\cos\lambda}{\mu}\right), \tag{A.9}$$

where c_1 is an integration constant.

The inverse Fourier transform of (A.9) with respect to t is

$$a(t) = \frac{c_1}{2\pi} \int_{-\infty}^{\infty} \exp\left[i\lambda \left(t + \frac{\zeta}{\mu} \right) - \frac{2\cos\lambda}{\mu} \right] d\lambda. \tag{A.10}$$

Setting $\tilde{\lambda} = \lambda + \pi/2$ in (A.10), noting $\zeta/\mu = -\overline{\omega} = -\omega/\Omega$, $1/\mu = i\beta/(2k\Omega)$ we obtain

$$a(t) = \frac{c_1}{2\pi} \exp\left[-i\frac{\pi}{2}\left(t + \frac{\zeta}{\mu} \right) \right] \int_{-\infty}^{\infty} \exp\left[i\tilde{\lambda}\left(t + \frac{\zeta}{\mu} \right) - \frac{2\sin\tilde{\lambda}}{\mu} \right] d\tilde{\lambda}$$

$$\equiv \frac{c_1}{2\pi} \exp\left[-i\frac{\pi}{2}(t - \overline{\omega}) \right] \int_{-\infty}^{\infty} \exp\left[i\tilde{\lambda}(t - \overline{\omega}) - \frac{i\beta}{k\Omega}\sin\tilde{\lambda} \right] d\tilde{\lambda}. \tag{A.11}$$

Setting $\hat{\lambda} = -\tilde{\lambda}$, (A.11) reduces to

$$a(t) = \frac{c_1}{2\pi} \exp\left[-i\frac{\pi}{2}(t - \overline{\omega}) \right] \int_{-\infty}^{\infty} \exp\left[-i\hat{\lambda}(t - \overline{\omega}) \right] \exp\left(\frac{i\beta}{k\Omega}\sin\hat{\lambda} \right) d\hat{\lambda}. \tag{A.12}$$

Using the Bessel function identity

$$\exp(ia\sin\theta) = \sum_{m=-\infty}^{\infty} \exp(im\theta) J_m(a) \tag{A.13}$$

(Abramowitz and Stegun 1965, formula 9.1.41, p. 361) in (A.12) and using the δ-function distribution representation

$$\delta(x' - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[ik(x' - x) \right] dk, \tag{A.14}$$

the solution (A.12) for $a(t)$ reduces to

$$a(t) = c_1 \exp\left[-i\frac{\pi}{2}(t - \overline{\omega}) \right] \sum_{m=-\infty}^{\infty} \delta(\overline{\omega} - t + m) J_m(a), \tag{A.15}$$

where

$$a = \frac{\beta}{k\Omega} \qquad \text{and} \qquad \overline{\omega} = \frac{\omega}{\Omega}. \tag{A.16}$$

Setting $t = n$ in (A.15) we obtain

$$a_n = c_1 \exp\left[-i\frac{\pi}{2}(n - \overline{\omega}) \right] J_{n-\overline{\omega}}(a), \tag{A.17}$$

which is the solution for a_n^1 in (35) for the case $c_1 = 1$.

The solution for a_n^2 in (35) can be derived by replacing λ by $-\lambda$ in the definition of the Fourier transform (A.4).