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Complete group classification of systems of two linear second-order ordinary differential equations: the algebraic approach

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We give a complete group classification of the general case of linear systems of two second-order ordinary differential equations. The algebraic approach is used to solve the group classification problem for this class of equations. This completes the results in the literature on the group classification of two linear second-order ordinary differential equations including recent results which give a complete group classification treatment of such systems. We show that using the algebraic approach leads to the study of a variety of cases in addition to those already obtained in the literature. We illustrate that this approach can be used as a useful tool in the group classification of this class of equations. A discussion of the subsequent cases and results is given. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

Systems of second-order ordinary differential equations appear in the modeling of many physical phenomena. A main feature of these differential equations is their symmetry properties. The theory of group analysis has been well studied in the literature. The presence of symmetries allows one to reduce the order of these equations or even find their general solution in quadratures.

Linear equations play a significant role among all ordinary differential equations: they are considered as a first approximation of the model being studied. In applications, linear equations often occur in disguised forms. In the study of the symmetries, it is convenient to rewrite the equations in their simplest equivalent form. We note that equations equivalent with respect to a change of the dependent and independent variables possess similar symmetry properties. This leads to a classification problem.

Systems of two linear second-order ordinary differential equations were studied in [1] where a new canonical form,

$$y'' = a(x)y + b(x)z,$$
$$z'' = c(x)y - a(x)z$$

was obtained. For this canonical form, the number of arbitrary elements is reduced. The group classification problem is usually simpler after reducing the number of arbitrary elements. In this paper, the authors also gave a representation of several admitted Lie groups. In addition, it was also proved that a system of two linear second-order ordinary differential equations can have 5, 6, 7, 8 or 15 point symmetries. However, the exhaustive list of all distinguished representatives of systems of two linear second-order ordinary differential equations was not obtained there.

The main objective of this paper is to use the algebraic approach where the determining equations presented in [2] are solved up to finding relations between constants defining admitted generators.

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The algebraic approach takes into account algebraic properties of Lie groups admitted by a system of equations: the knowledge of the algebraic structure of admitted Lie algebras allows for significant simplification of the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S. Lie [3, 4], cannot be performed without using the algebraic structure of the admitted Lie groups. Recently, the algebraic properties were applied for group classification, for example, in [5–12]. We also note that the use of the algebraic structure of admitted Lie groups completely simplified the group classification of equations describing behavior of fluids with internal inertia in [13].

In the present paper, we obtain a complete group classification of the general case of linear systems of two second-order ordinary differential equations

$$y'' = F(x, y, z), \quad z'' = G(x, y, z),$$

by using an algebraic approach. The system considered in this case is a generalization of Lie's study [4]. Excluded from our consideration is the studied earlier systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{1}$$

where M is a matrix with constant entries and $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$. These cases of systems have been studied in [14–18]. We also exclude from the analysis the degenerate case given as follows:

$$y'' = F(x, y, z),\ z'' = 0. \tag{2}$$

It is worth mentioning here that the complete group classification of two linear second-order ordinary differential equations has been done recently in [2]. The following four cases of linear systems of equations with none inconstant coefficients were obtained:

$$F = \alpha_{11}y + e^x z, \qquad G = e^{-x}\alpha_{21}y + \alpha_{22}z, \tag{3}$$

$$F = y(\sin(x) + c_2) + z(\cos(x) - c_1), \qquad G = y(\cos(x) + c_1) + z(-\sin(x) + c_2), \tag{4}$$

$$F = y(\alpha_{11} + x) + z(\alpha_{12} + (\alpha_{22} - \alpha_{11})x - x^2), \qquad G = y + z(-x + \alpha_{22}), \tag{5}$$

$$F = yc + z, \qquad G = -y + zc \tag{6}$$

where α_{ij} and c_i $(i, j = 1, 2)$ are constant, $c = c(x)$ and $\alpha_{21}c' \neq 0$. These systems have the following nontrivial admitted generators:

System	Admitted generator
(3)	$\partial_x - z\partial_z$
(4)	$2\partial_x + z\partial_y - y\partial_z$
(5)	$\partial_x + z\partial_y$
(6)	$z\partial_y - y\partial_z$

We note that the approach used in [2] is different from the approach used in the present paper. Because there is an opinion that the algebraic approach is more efficient, the present paper can be a good example for comparing these two approaches. We show here that for the problem of classification of systems of two linear second-order ordinary differential equations the algebraic approach leads to the study of a variety of cases, although the analysis of these cases is not complicated.

The paper is organized as follows. The first part of the paper deals with the preliminary study of systems of two second-order linear equations followed by the group classification method as applied to linear systems of equations. The subsequent subsections deal with the equivalence transformations, determining equations and the optimal system of subalgebras. The later part lists the different cases with their respective results. This is then followed by the results and conclusion.

2. Preliminary study of systems of linear equations

Linear second-order ordinary differential equations have the following form,

$$\mathbf{y}'' = B(x)\mathbf{y}' + A(x)\mathbf{y} + f(x), \tag{7}$$

where $A(x)$ and $B(x)$ are $n \times n$ matrices, and $f(x)$ is a vector. Using a particular solution $\mathbf{y}_p(x)$ and the change of variable,

$$\mathbf{y} = \tilde{\mathbf{y}} + \mathbf{y}_p,$$

one can without loss of generality assume that $f(x) = 0$. Applying the change

$$\mathbf{y} = C(x)\bar{\mathbf{y}},$$

where $C = C(x)$ is a nonsingular matrix, system (7) becomes

$$\bar{\mathbf{y}}'' = \bar{B}\bar{\mathbf{y}}' + \bar{A}\bar{\mathbf{y}}, \tag{8}$$

where

$$\bar{B} = C^{-1}(BC - 2C'), \quad \bar{A} = C^{-1}(AC + BC' - C'').$$

If one chooses the matrix $C(x)$ such that

$$C' = \frac{1}{2}BC,$$

then

$$\bar{B} = 0, \quad \bar{A} = C^{-1}\left(A + \frac{1}{4}B^2 - \frac{1}{2}B'\right)C. \tag{9}$$

The existence of the nonsingular matrix $C(x)$ is guaranteed by the existence of the solution of the Cauchy problem

$$C' = \frac{1}{2}BC, \quad C(0) = E,$$

where E is the unit $n \times n$ matrix.

Notice that if the matrices A and B are constant, then the matrix $\bar{A}$ in (9) is constant only for commuting matrices A and B. The complete study of noncommutative constant matrices A and B was done recently in [19].

Without loss of generality up to equivalence transformations in the class of systems of the form (7), it suffices to study the systems of the form

$$\mathbf{y}'' = A\mathbf{y}. \tag{10}$$

Applying the change of the dependent and independent variables [2]

$$\tilde{x} = \varphi(x), \quad \tilde{\mathbf{y}} = \psi(x)\mathbf{y} \tag{11}$$

satisfying the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}, \tag{12}$$

system (10) becomes

$$\tilde{\mathbf{y}}'' = \tilde{A}\tilde{\mathbf{y}}, \tag{13}$$

where

$$\tilde{A} = \varphi'^{-2}\left(A - \frac{\rho''}{\rho}E\right), \quad \rho = \frac{1}{\psi}.$$

For reducing the number of entries of the matrix $\tilde{A}$, one can choose the function ψ such that[‡] $tr\,\tilde{A} = 0$. This condition leads to the equation

$$\rho'' - \frac{tr\,A}{n}\rho = 0. \tag{14}$$

Notice that in particular, for matrices with $tr\,A = 0$ choosing $\rho = c_1x + c_2$, the matrix $\tilde{A}$ still satisfies the condition $tr\,\tilde{A} = 0$. Here,

$$\psi = (c_1x + c_2)^{-1}, \quad \varphi' = k_0\psi^2 = k_0(c_1x + c_2)^{-2}, \tag{15}$$

where k_0 is constant.

[‡] *This change was used in* [1] *for the case of* $n = 2$.

3. Group classification

For the group classification of systems of two linear second-order ordinary differential equations, we consider a system of equations (10) with a matrix

$$A = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix}.$$

Any linear system of ordinary differential equations (10) admits the following trivial generators

$$y\partial_y + z\partial_z, \tag{16}$$

$$h(x)\partial_y + g(x)\partial_z, \tag{17}$$

where (16) is the homogeneity symmetry and $(h(x), g(x))^t$ is any solution of system (10).

For the classification problem one needs to study equations which admit generators different from (16) and (17).

3.1. *Equivalence transformations*

Calculations show that the equivalence Lie group is defined by the generators:

$$\begin{array}{l} X_1^e = x(x\partial_x + y\partial_y + z\partial_z - 4a\partial_a - 4b\partial_b - 4c\partial_c), \\ X_2^e = 2x\partial_x + y\partial_y + z\partial_z - 4a\partial_a - 4b\partial_b - 4c\partial_c, \\ X_3^e = \partial_x, \quad X_4^e = y\partial_z - b\partial_a + 2a\partial_c, \quad X_5^e = z\partial_y + c\partial_a - 2a\partial_b, \\ X_6^e = y\partial_y - z\partial_z - 2b\partial_b + 2c\partial_c, \quad X_7^e = y\partial_y + z\partial_z. \end{array}$$

The transformations corresponding to the generator X_1^e define transformations of the form (15). The transformation corresponding to X_2^e and X_3^e define the dilation and shift of x, respectively. The transformations related with the generators X_4^e, X_5^e, X_6^e and X_7^e correspond to the linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix P.

3.2. *Determining equations*

According to the Lie algorithm [20], the generator

$$X = \xi(x,y,z)\frac{\partial}{\partial x} + \eta_1(x,y,z)\frac{\partial}{\partial y} + \eta_2(x,y,z)\frac{\partial}{\partial z}$$

is admitted by system (10) if it satisfies the associated determining equations. One can show that the admitted generator has the property that $\xi_y^2 + \xi_z^2 \neq 0$ if and only if system (10) is equivalent to the free particle equations [2]. Hence, one obtains $\xi = \xi(x)$. The determining equations are

$$b(\xi' z + zq_4 + yq_3) + a(\xi' y + zq_2 + yq_1) + 2(a'y + b'z)\xi - \xi''' y + (3\xi' - q_1)(ay + bz) - q_1(cy - az) = 0,$$

and

$$-a(\xi' z + zq_4 + yq_3) + c(\xi' y + zq_1 + yq_2) + 2(c'y - a'z)\xi - \xi''' z - q_3(ay + bz) + (3\xi' - q_4)(cy - az) = 0,$$

where an admitted generator has the form

$$X = 2\xi(x)\partial_x + (y\xi'(x) + q_1 y + q_2 z)\partial_y + (\xi' z + q_3 y + q_4 z)\partial_z$$

and q_i, $(i = 1, .., 4)$ are constant. We exclude the trivial admitted generators (17).

Splitting the determining equations with respect to y and z leads to $\xi''' = 0$ and the equations[§]

$$2a'\xi + 4a\xi' + bq_3 - cq_1 = 0, \tag{18}$$

$$2b'\xi + 2aq_1 + b(4\xi' + q_4 - q_2) = 0, \tag{19}$$

[§] *These equations coincide with equations (32)–(34) of* [1], *where the constants from* [1] *are* $s_0 = q_1$, $r_0 = q_2$, $p_0 = q_3$, $q_0 = q_4$. *The difference is that in our study, there is no necessity at this stage of the assumption* $b \neq 0$ *comparing with* [1].

$$2c'\xi - 2aq_3 + c(4\xi' - q_4 + q_2) = 0. \qquad (20)$$

Because $\xi^{(3)} = 0$ or $\xi = a_1x^2 + a_2x + a_3$, the admitted generators have the form

$$X = a_1X_1 + a_2X_2 + a_3X_3 + q_3X_4 + q_1X_5 + \frac{q_2 - q_4}{2}X_6 + \frac{q_2 + q_4}{2}X_7,$$

where

$$X_1 = x(x\partial_x + y\partial_y + z\partial_z),\ X_2 = 2x\partial_x + y\partial_y + z\partial_z,\ X_3 = \partial_x,$$

$$X_4 = y\partial_z,\ X_5 = z\partial_y,\ X_6 = y\partial_y - z\partial_z,\ X_7 = y\partial_y + z\partial_z.$$

In addition, because the generator X_7 is the trivial admitted generator (16), one can assume that $q_4 = -q_2$. The constants a_1, a_2, a_3, q_1, q_2 and q_3 depend on the functions $a(x), b(x)$ and $c(x)$. These relations are defined by equations (18)–(20), and they present the group classification of linear systems of two second-order ordinary differential equations.

One of the methods for analyzing relations between the constants consists of employing the algorithm developed for the gas dynamics equations [20]. This algorithm allows one to study all possible admitted Lie algebras without omission. Unfortunately, it is difficult to implement for system (10). Observe also that in this approach it is difficult to select out equivalent cases with respect to equivalence transformations.

In [9, 11, 12][¶] a different approach was applied for the group classification. In most applications the algebraic algorithm essentially reduces the study of group classification to a simpler problem. Here, we follow this approach.

For further analysis we study the Lie algebra L_6 spanned by the generators $X_1, X_2, \ldots, X_6$.

3.3. Optimal system of subalgebras of L_6

The Lie algebra $L_6 = L_3^{(1)} \oplus L_3^{(2)}$, where $L_3^{(1)} = \{X_1, X_2, X_3\}$, $L_3^{(2)} = \{X_4, X_5, X_6\}$. The commutator table can be split into two tables:

	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

	X_4	X_5	X_6
X_4	0	X_6	$-2X_4$
X_5	$-X_6$	0	$2X_5$
X_6	$2X_4$	$-2X_5$	0.

Denoting

$$X_1 = e_1, \quad X_2 = -2e_2, \quad X_3 = e_3,$$

one can show that the commutator table of the Lie algebra $L_3^{(1)}$ becomes

	e_1	e_2	e_3
e_1	0	e_1	$2e_2$
e_2	$-2e_2$	0	e_3
e_3	$-2e_2$	$-e_3$	0

.

Hence, the Lie algebra[‖] $L_3^{(1)}$ is sl(2, R). One also can check that $L_3^{(2)}$ is sl(2, R) by denoting

$$X_4 = -e_1, \quad X_5 = e_3, \quad X_6 = -2e_2.$$

Notice that an optimal system of subalgebras of the Lie algebra sl(2, R) classification was performed in [21] and it consists of the following list:

$$\{e_2\}, \quad \{e_3\}, \quad \{e_1 + e_3\}, \quad \{e_2, e_3\}, \quad \{e_1, e_2, e_3\}. \qquad (21)$$

Then, the optimal systems of subalgebras of $L_3^{(1)}$ and $L_3^{(2)}$ are

$$\{X_2\}, \quad \{X_3\}, \quad \{X_1 + X_3\}, \quad \{X_2, X_3\}, \quad \{X_1, X_2, X_3\}, \qquad (22)$$

and

$$\{X_5\}, \quad \{X_6\}, \quad \{X_4 - X_5\}, \quad \{X_5, X_6\}, \quad \{X_4, X_5, X_6\}, \qquad (23)$$

respectively.

¶ *See also references therein.*

‖ *This Lie algebra is a Lie algebra of type VIII in the Bianchi classification.*

3.4. Relations between automorphisms and equivalence transformations

Let us consider an operator

$$X = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6.$$

Automorphisms of the Lie algebra L_6 are

$$\begin{array}{ll} A_1: & \bar{x}_1 = x_1 + 2ax_2 + a^2x_3,\ \bar{x}_2 = x_2 + ax_3; \\ A_2: & \bar{x}_1 = x_1e^a,\ \bar{x}_3 = x_3e^{-a}; \\ A_3: & \bar{x}_2 = x_2 + ax_1,\ \bar{x}_3 = x_3 + 2ax_2 + a^2x_1, \\ A_4: & \bar{x}_4 = x_4 - 2ax_6 - a^2x_5,\ \bar{x}_6 = x_6 + ax_5; \\ A_5: & \bar{x}_5 = x_5 + 2ax_6 - a^2x_4,\ \bar{x}_6 = x_6 - ax_4; \\ A_6: & \bar{x}_4 = x_4e^a,\ \bar{x}_5 = x_5e^{-a}. \end{array}$$

Here and further on, only changeable coordinates are presented.

One can show that actions of equivalence transformations are similar to actions of the automorphisms. These properties allow one to use an optimal system of subalgebras of the Lie algebra L_6 for group classification.

In fact, using the change of the dependent and independent variables corresponding to the equivalence transformation (11) with

$$\varphi = \frac{x}{1-\tau x}, \quad \psi = (x+\tau)^{-1},$$

the operator

$$X = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6$$

becomes

$$X = \bar{x}_1\bar{X}_1 + \bar{x}_2\bar{X}_2 + \bar{x}_3\bar{X}_3 + \bar{x}_4\bar{X}_4 + \bar{x}_5\bar{X}_5 + \bar{x}_6\bar{X}_6,$$

where

$$\bar{X}_1 = \bar{x}(\bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}}),\ \bar{X}_2 = 2\bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}},\ \bar{X}_3 = \partial_{\bar{x}},$$
$$\bar{X}_4 = \bar{y}\partial_{\bar{z}},\ \bar{X}_5 = \bar{z}\partial_{\bar{y}},\ \bar{X}_6 = \bar{y}\partial_{\bar{y}} - \bar{z}\partial_{\bar{z}},$$

and the changeable coefficients are

$$\bar{x}_1 = x_1 + 2x_2\tau + x_3\tau^2,\ \bar{x}_2 = x_2 + x_1\tau.$$

Hence, the change of the dependent and independent variables corresponding to the equivalence transformation X_1^e is similar to the automorphism Aut_1. This property we denote as $X_1^e \sim Aut_1$. Similarly, one can check that $X_i^e \sim Aut_i$, $(i = 2, 3, \dots, 6)$.

Using the two-step algorithm of constructing an optimal system of subalgebras [22], and the optimal systems of subalgebras (22) and (23), one obtains an optimal system of one-dimensional subalgebras of the Lie algebra L_6 which consists of the following set of subalgebras

1.1.	$X_2 + \gamma(X_4 - X_5)$	3.1.	$X_1 + X_3 + \gamma(X_4 - X_5)$
1.2.	$X_2 + \gamma X_5$	3.2.	$X_1 + X_3 + \gamma X_5$
1.3.	$X_2 + \gamma X_6$	3.3.	$X_1 + X_3 + \gamma X_6$
2.1.	$X_3 + \gamma(X_4 - X_5)$	4.1.	$X_4 - X_5$
2.2.	$X_3 + \gamma X_5$	4.2.	X_5
2.3.	$X_3 + \gamma X_6$	4.3.	X_6

where γ is an arbitrary constant.

4. Solutions of the determining equations

We obtained the condition that for the group classification, one needs to construct solutions of equations (18)–(20), where the constants are

$$a_1 = x_1,\ a_2 = x_2,\ a_3 = x_3,\ q_3 = x_4,\ q_1 = x_5,\ q_2 = x_6,\ q_4 = -x_6.$$

Here, x_i $(i = 1, 2, \dots, 7)$ are coordinates of the generator

$$X = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6$$

chosen from the optimal system of subalgebras.

Notice that the subalgebra with the generator X_3 corresponds to equations with constant coefficients. One can also check that using equivalence transformation (11) with a corresponding function $\bar{x} = \psi(x)$, the generators presented in the optimal system of subalgebras for $\gamma = 0$ are reduced to the generator $\bar{X}_3 = \partial_{\bar{x}}$. Hence, we only need to consider the cases where $\gamma \neq 0$.

4.1. *Subalgebra 1.1 with the generator* $X_2 + \gamma(X_4 - X_5)$

In this case, equations (18)–(20) become

$$\begin{aligned} 2xa' + 4a + \gamma(b+c) &= 0, \\ xb' + 2b - \gamma a &= 0, \\ xc' + 2c - \gamma a &= 0. \end{aligned} \tag{24}$$

Applying the change

$$a = x^{-2}\bar{a}, \;\; b = x^{-2}\bar{b}, \;\; c = x^{-2}\bar{c},$$

equation (24) is reduced to the equations

$$\begin{aligned} 2x\bar{a}' + \gamma(\bar{b}+\bar{c}) &= 0, \\ x\bar{b}' - \gamma\bar{a} &= 0, \\ x\bar{c}' - \gamma\bar{a} &= 0. \end{aligned} \tag{25}$$

From the last two equations of (25), one finds that $\bar{b} = \bar{c} + k$, where k is a constant. Denoting $\bar{c} = \tilde{c} - k/2$, the remaining equations of (25) become

$$\begin{aligned} x\bar{a}' + \gamma\tilde{c} &= 0, \\ x\tilde{c}' - \gamma\bar{a} &= 0. \end{aligned} \tag{26}$$

Then, from the second equation of (26), one finds

$$\bar{a} = \gamma^{-1}x\tilde{c}'.$$

The first equation of (26) becomes

$$x^2\bar{a}'' + x\bar{a}' - \gamma^2\bar{a} = 0.$$

This is the Euler type equation with general solution

$$\bar{a} = k_1\sin(\gamma\ln x) + k_2\cos(\gamma\ln x).$$

Finally, we obtain

$$a = \frac{k_1\sin(\gamma\ln x) + k_2\cos(\gamma\ln x)}{x^2}, \;\; b = \frac{k - 2k_1\cos(\gamma\ln x) + 2k_2\sin(\gamma\ln x)}{2x^2},$$
$$c = \frac{-k - 2k_1\cos(\gamma\ln x) + 2k_2\sin(\gamma\ln x)}{2x^2}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (4).

4.2. *Subalgebra 1.2 with the generator* $X_2 + \gamma X_5$

Equations (18)–(20) in this case are reduced to

$$\begin{aligned} 2xa' + 4a - \gamma c &= 0 \\ xb' + 4b + \gamma a &= 0 \\ xc' + 2c &= 0. \end{aligned} \tag{27}$$

Applying the change

$$a = x^{-2}\bar{a}, \;\; b = x^{-2}\bar{b}, \;\; c = x^{-2}\bar{c},$$

equations (27) become

$$\begin{aligned} 2x\bar{a}' - \gamma k &= 0 \\ x\bar{b}' + \gamma\bar{a} &= 0, \end{aligned} \tag{28}$$

where $\bar{c} = k$, k is a constant.

From the first equation of (28), we have

$$\bar{a}' = \frac{\gamma k}{2x}.$$

From the second equation of (28), one obtains

$$x^2\bar{b}'' + x\bar{b}' + \frac{\gamma^2 k}{2} = 0 \tag{29}$$

of which the general solution is

$$\bar{b} = -\frac{\gamma^2 k}{4}(\ln x)^2 + k_2 \ln x + k_3.$$

Finally,

$$a = \frac{k_1 + \gamma k \ln x}{2x^2}, \quad b = \frac{k_3 + 4k_2 \ln x - \gamma^2 k(\ln x)^2}{4x^2}, \quad c = \frac{k}{x^2}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (5).

4.3. Subalgebra 1.3 with the generator $X_2 + \gamma X_6$

Equations (18)–(20) can be expressed in this form

$$\begin{aligned} xa' + 2a &= 0 \\ xb' + (2-\gamma)b &= 0 \\ xc' + (\gamma+2)c &= 0. \end{aligned} \tag{30}$$

Solving (30), one obtains

$$a = \frac{k_1}{x^2}, \quad b = \frac{k_2}{x^{2-\gamma}}, \quad c = \frac{k_3}{x^{2+\gamma}}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (3).

4.4. Subalgebra 2.1 with the generator $X_3 + \gamma(X_4 - X_5)$

In this case, we have

$$\begin{aligned} 2a' + \gamma(b+c) &= 0 \\ b' - \gamma a &= 0 \\ c' - \gamma a &= 0. \end{aligned} \tag{31}$$

From equations (31), we get

$$a'' + \gamma^2 a = 0. \tag{32}$$

Then, the solution is

$$a = k_1 \sin(\gamma x) + k_2 \cos(\gamma x), \quad b = k_1 \gamma x \sin(\gamma x) + k_2 \gamma x \cos(\gamma x) + k,$$

$$c = k_1 \gamma x \sin(\gamma x) + k_2 \gamma x \cos(\gamma x) - k,$$

where k_1 and k_2 are constant.

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (4).

4.5. *Subalgebra 2.2 with the generator $X_3 + \gamma X_5$*

We can write equations (18)–(20) as

$$\begin{aligned} 2a' - \gamma c &= 0 \\ b' - \gamma a &= 0 \\ c' &= 0. \end{aligned} \tag{33}$$

From (33), we have

$$b'' = \frac{\gamma^2 k}{2},$$

where $c = k$, k is the constant.

Therefore,

$$a = \frac{\gamma k}{2}x + k_1, \;\; b = \frac{\gamma^2 k}{4}x^2 + k_1 x + k_2,$$

where k_1 and k_2 are constant.

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (5).

4.6. *Subalgebra 2.3 with the generator $X_3 + \gamma X_6$*

From equations (18)–(20), one obtains

$$\begin{aligned} a' &= 0 \\ b' - \gamma b &= 0 \\ c' + \gamma c &= 0. \end{aligned} \tag{34}$$

Therefore, the solutions to (34) are

$$a = k, \;\; b = be^{\gamma x}, \;\; c = ke^{-\gamma x}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (3).

4.7. *Subalgebra 3.1 with the generator $X_1 + X_3 + \gamma(X_4 - X_5)$*

In this case, equations (18)–(20) become

$$\begin{aligned} 2a'(x^2+1) + 8ax + \gamma(b+c) &= 0 \\ b'(x^2+1) - \gamma a + 4bx &= 0 \\ c'(x^2+1) - \gamma a + 4cx &= 0 \end{aligned} \tag{35}$$

where $\alpha = \pm 1$. We solve (35) by applying the change

$$a = (x^2+1)^{-2}\bar{a}, \;\; b = (x^2+1)^{-2}\bar{b}, \;\; c = (x^2+1)^{-2}\bar{c},$$

and thus, we obtain

$$\begin{aligned} (x^2+1)\bar{a}' + \frac{\gamma}{2}(\bar{b}+\bar{c}) &= 0 \\ (x^2+1)\bar{b}' - \gamma\bar{a} &= 0 \\ (x^2+1)\bar{c}' - \gamma\bar{a} &= 0. \end{aligned} \tag{36}$$

From equations (36)

$$\bar{c} = \bar{b} + k,$$

where k is a constant and

$$(x^2+1)\bar{a}' + \frac{\gamma}{2}(2\bar{b} + k) = 0.$$

Therefore,

$$\bar{b}' = \frac{\gamma \bar{a}}{x^2 + 1}.$$

Hence,

$$(x^2 + 1)^2 \bar{a}'' + 2x(x^2 + 1)\bar{a}' + \gamma^2 \bar{a} = 0 \tag{37}$$

and equation (37) is reduced to

$$\frac{d^2 \bar{a}}{d\bar{x}^2} + \gamma^2 \bar{a} = 0$$

with solution

$$\bar{a} = k_1 \cos(\gamma \arctan x) + k_2 \sin(\gamma \arctan x).$$

Therefore,

$$\begin{aligned}
a &= (x^2 + 1)^{-2} \left(k_1 \cos(\gamma \arctan x) + k_2 \sin(\gamma \arctan x)\right), \\
b &= (x^2 + 1)^{-2} \left(2k_1 \sin(\gamma \arctan x) - 2k_2 \cos(\gamma \arctan x) + k\right), \\
c &= (x^2 + 1)^{-2} \left(2k_1 \sin(\gamma \arctan x) - 2k_2 \cos(\gamma \arctan x) - k\right).
\end{aligned}$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (4).

4.8. *Subalgebra 3.2 with the generator* $X_1 + X_3 + \gamma X_5$

In this case, equations (18)–(20) become

$$\begin{aligned}
2a'(x^2 + 1) + 8ax - \gamma c &= 0, \\
b'(x^2 + 1) + 4bx + \gamma a &= 0, \\
c'(x^2 + 1) + 4cx &= 0.
\end{aligned} \tag{38}$$

Applying the change

$$a = (x^2 + 1)^{-2}\bar{a}, \;\; b = (x^2 + 1)^{-2}\bar{b}, \;\; c = (x^2 + 1)^{-2}\bar{c},$$

equations (38) are reduced to the equations

$$\begin{aligned}
(x^2 + 1)\bar{a}' - \frac{\gamma}{2}k &= 0, \\
(x^2 + 1)\bar{b}' + \gamma \bar{a} &= 0,
\end{aligned} \tag{39}$$

where $\bar{c} = k$, k is a constant. From the first equation of (39)

$$\bar{a}' = \frac{\gamma k}{2(x^2 + 1)}$$

or

$$a = \frac{\gamma k \arctan\, x + k_1}{(x^2 + 1)^2},$$

and the second equation of (39) becomes

$$\bar{b}' = -\gamma \left(k_1 + \frac{\gamma k}{2}\bar{x}\right), \tag{40}$$

where $\bar{x} = \arctan\, x$. The solution of this equation is

$$\bar{b} = -\frac{\gamma^2 k}{4}(\arctan\, x)^2 - \gamma k_1 \arctan\, x + k_2.$$

Finally,

$$a = \frac{\gamma k \arctan x + k_1}{2(x^2+1)^2}, \quad b = -\frac{\gamma^2 k(\arctan x)^2 + 4\gamma k_1 \arctan x + k_2}{4(x^2+1)^2}, \quad c = \frac{k}{(x^2+1)^2}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (5).

4.9. Subalgebra 3.3 with the generator $X_1 + X_3 + \gamma X_6$

In this case, equations (18)–(20) become

$$\begin{aligned} a'(x^2+1) + 4ax &= 0, \\ b'(x^2+1) + b(4x-\gamma) &= 0, \\ c'(x^2+1) + c(4x+\gamma) &= 0. \end{aligned}$$

The general solution of equations (41) is

$$a = \frac{k_1}{(x^2+1)^2}, \quad b = \frac{k_2}{(x^2+1)^2} e^{\gamma \arctan x}, \quad c = \frac{k_3}{(x^2+1)^2} e^{-\gamma \arctan x}.$$

We note that this case of equations (10) can be reduced by a point transformation to the equations with arbitrary elements of the form (3).

4.10. Subalgebra 4.1 with the generator $X_4 - X_5$

In this case, we solve equations (18)–(20) to obtain

$$a = 0, \quad b = -c.$$

This case of equations of (10) with $c' \neq 0$ belongs to the class of equations of the form (6).

4.11. Subalgebra 4.2 with the generator X_5

In this case, we solve equations (18)–(20) to obtain

$$a = 0, \quad c = 0.$$

In this case, the second equation of (10) is reduced to the free particle equation. This case is excluded from our consideration.

4.12. Subalgebra 4.3 with the generator X_5

In this case, we solve equations (18)–(20) to obtain

$$b = 0, \quad c = 0.$$

This case is also excluded from our consideration.

5. Discussion on solving the determining equations

We note that the linear combination, where equation (18) is multiplied by $q_2 - q_4$, equation (19) is multiplied by q_3, and equation (20) is multiplied by q_1 gives the integral

$$(h\xi^2)' = 0,$$

where $h = (q_2 - q_4)a + q_3 b + q_1 c$. In particular, for $\xi \neq 0$, this gives

$$(q_2 - q_4)a + q_3 b + q_1 c = k\xi^{-2},$$

where k is constant. Moreover, in this case, the change

$$\bar{x} = \varphi(x), \quad \bar{a}\xi^{-2}, \quad b = \bar{b}\xi^{-2}, \quad c = \bar{c}\xi^{-2},$$

where $2\varphi'\xi = 1$ reduces equations (18)–(20) to the simpler form

$$\frac{d\bar{a}}{d\bar{x}} + \bar{b}q_3 - \bar{c}q_1 = 0,$$
$$\frac{d\bar{b}}{d\bar{x}} + 2(\bar{a}q_1 - \bar{b}q_2) = 0,$$
$$\frac{d\bar{c}}{d\bar{x}} - 2(\bar{a}q_3 - q_2\bar{c}) = 0.$$

The latter system can be rewritten in the matrix form

$$\frac{d}{d\bar{x}}\bar{A} + \bar{A}H - H\bar{A} = 0,$$

where

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix}, \quad H = \begin{pmatrix} q_2 & q_1 \\ q_3 & -q_2 \end{pmatrix}.$$

The general solution of the matrix equation is [23]

$$\bar{A} = e^{\bar{x}H} A_0 e^{-\bar{x}H},$$

where the matrix

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix}$$

is a matrix with arbitrary constant entries a_0, b_0 and c_0. The following three particular cases of the matrix H are used earlier:

$$H_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For these matrices, their corresponding exponential matrices e^{sH_i} are

$$e^{sH_1} = E + sH_1 - \frac{s^2}{2!}E - \frac{s^3}{3!}H_1 + \cdots = \cos(s)E + \sin(s)H_1,$$
$$e^{sH_2} = E + sH_2, \quad e^{sH_3} = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}.$$

Notice also that for $\gamma = 0$ the matrix $H = 0$, which means that the matrix $\bar{A}$ is constant.

6. Algebras of dimensions $n \geq 2$

Assuming that in the admitted $n \geq 2$ dimensional Lie algebra there exists one generator such that $\xi = 0$, one finds that this generator has to be $X_4 - X_5$ and the system is with $a = 0$ and $c = -b \neq 0$. Substituting these values into (18)–(20), one finds that other generators can be written in the form

$$x_1X_1 + x_2X_2 + x_3X_3, \quad (x_1^2 + x_2^2 + x_3^2 \neq 0).$$

As shown earlier, systems (10) admitting such generators are equivalent to a system with constant coefficients.

Assuming that in the admitted Lie algebra there are two linearly independent generators with $\xi \neq 0$, one can conclude that a set of basis generators contains the generators

$$X_2 + x_4X_4 + x_5X_5 + x_6X_6, \quad X_3 + k(y_4X_4 + y_5X_5 + y_6X_6),$$

where k is some constant chosen for simplicity as will be explained further. Notice also that because for $k = 0$ the matrix A is constant, one has to assume that $k \neq 0$.

Substituting the coefficients of the second generator into system (18)–(20), where the coefficients y_4, y_5 and y_6 are chosen from the optimal system (23), one finds the derivatives a', b' and c'. After the next substitution of the coefficients of the first generator into system (18)–(20), from equation (18) we obtain

$$a = f_1 b + f_2 c,$$

where $f_i(x)$ are some functions. The remaining equations (18)–(20) compose a system of two algebraic linear homogeneous equations with respect to b and c. If the determinant of this system $\Delta(x)$ is not equal to zero, then $b = 0, c = 0$ and $a = 0$. Hence, one needs to study the case where $\Delta(x) = 0$. Because $\Delta(x)$ is a polynomial with respect to x, one can split it. The splitting leads to the case where $k = 0$.

Thus, there are no systems of equations (10), admitting more than one nontrivial generator, which are not equivalent to a constant-coefficient system (1).

7. Conclusion

We have given a complete group classification of the general case of linear systems of two second-order ordinary differential equations excluding the systems which are equivalent to systems of the type (1) and the degenerate case (2) using the algebraic approach. We were able to apply the algebraic approach because the study is reduced to the analysis of relations between constants. The cause of this possibility is the property $tr\,A = 0$. This condition led us to the equation $\xi^{(3)} = 0$. A complete group classification of the general case is obtained. The delineated list obtained further shows that the problem of classification of systems of two linear second-order ordinary differential equations using the algebraic approach leads to the study of a variety of cases, and this approach can be used as an effective tool to study the group classification of the type of systems studied here. This adds to the body of knowledge in the literature on this subject including the recent results in [2].

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