

Analytical solutions in $R + qR^n$ cosmology from singularity analysis

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The integrability of higher-order theories of gravity is of importance in the determining the properties of these models and so their viability as models of reality. An important tool in the establishment of integrability is the singularity analysis. We apply this analysis to the case of fourth-order theory of gravity $f(R) = R + qR^n$ to establish those values of the free parameters q and n for which integrability in this sense exists. As a preliminary we examine the well-known case of $n = 4/3$.

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The generalization of the Einstein-Hilbert Action of General Relativity (GR), which leads to the so-called modified theories of gravity, has been proposed in order to explain some of the recent cosmological observations. The novelty of the modified theories of gravity is that the dark energy, which is related with the late-time acceleration of the universe, has its origins in the additional dynamical quantities which follow from the gravitational action [1].

A proposed generalization of the Einstein-Hilbert action is the so-called $f(R)$ -gravity in which the gravitational Action Integral, in a four-dimensional Riemannian manifold M with metric tensor $g_{\mu\nu}$ of Lorentzian signature, is given as follows [2]

$$S_{f(R)} = \int dx^4 \sqrt{-g} f(R) + S_m, \quad (1)$$

where R is the Ricci Scalar of the spacetime. The Action Integral, (1), includes that of GR with or without cosmological constant when $f(R)$ is a linear function, i.e., $f_{,RR} = 0$. Furthermore the Ricci Scalar admits second-order derivatives of the metric. However, when $f(R)$ is a linear function, integration by parts of (1) gives that the Lagrangian of the field equation admits only first derivatives, that is, the gravitational theory is a second-order gravity. However, the latter is not true for a nonlinear function, $f(R)$, where the theory is a fourth-order gravity¹.

It is well known that a fourth-order differential equation can be written as a system of two second-order differential equations with the use of a Lagrange Multiplier. The analogy in Classical Mechanics is the Legendre Transformation between the Euler-Lagrange equations and Hamilton's equations. Furthermore variation with respect to the metric tensor in (1) leads to the set of equations,

$$G_{\mu\nu}^{f(R)} \equiv f_{,R} R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\sigma \nabla^\sigma) f_{,R} = T_{\mu\nu}, \quad (2)$$

in which $T_{\mu\nu}$ is the energy-momentum tensor which is related with the matter source, S_m in (1), and ∇_μ is the covariant derivative associated with the Levi-Civita connection Riemannian manifold with metric tensor $g_{\mu\nu}$ and $R_{\mu\nu}$ is the Ricci tensor field. Easily we can see that for nonlinear function $f(R)$, the third term of the left hand side of (2) provides the fourth-order derivatives.

However, if we assume the Ricci Scalar to be one of the dynamical variables of the system, then only second-order derivatives exist in (2), while the new equation which we have to consider is $R = R^{\mu\nu} R_{\mu\nu}$. Another parametrization that has been proposed is to define the field $\phi = f_{,R}$, where the field equations, (2), correspond to those of Scalar-tensor theories and, specifically, to O'Hanlon gravity [3]. Furthermore the same results follow even if with a Lagrange Multiplier λ [6] and a new field R or ϕ in the gravitational action (1).

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¹ For reviews in $f(R)$ theories of gravity see [4, 5].

In the context of this work we are interested in the singularity analysis of the field equations, (2), at the cosmological level. We select to work on the reparameterization. in which the new dynamical variable that makes the field equations a system of second-order differential equations is the Ricci Scalar R . Furthermore we consider the metric $g_{\mu\nu}$ to be that of a Friedmann-Lemaître-Robertson-Walker spacetime (FLRW) with zero spatial curvature, which gives that the field equations (2) follow from the variational principle of the Lagrangian function,

$$L(N, a, \dot{a}, R, \dot{R}) = \frac{1}{N} \left(6af'\dot{a}^2 + 6a^2f''\dot{a}\dot{R} \right) + N \left(a^3(f'R - f) + 2\rho_{m0} \right), \quad (3)$$

where we assumed the matter source to be that of dust fluid, $a(t)$ is the scale-factor and $N(t)$ is the lapse-function; that is, the line element of the FLRW spacetime is

$$ds^2 = -N^2(t) dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \quad (4)$$

We select to work in the lapse in which $N(t) = 1$ where the comoving observer is $u^\mu = \delta_t^\mu$, ($u^\mu u_\mu = -1$). Therefore the field equations, (2), are²

$$3f'H^2 + 3Hf''\dot{R} - (f'R - f) = \frac{\rho_{m0}}{a^3} \quad (5)$$

and

$$2f'\dot{H} + 3f'H^2 = -2Hf''\dot{R} - \left(f'''\dot{R}^2 + f''\ddot{R} \right) - \frac{f - Rf'}{2}, \quad (6)$$

while the constraint equation $R = R^{\mu\nu}R_{\mu\nu}$ is

$$R = 6 \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right). \quad (7)$$

A useful observation for equations (5)-(7) is that they form a Hamiltonian system, in which (5) is the Hamiltonian function³. That means that the solution of the system, (6), (7), when it is substituted into (5), provides a constraint equation for the free parameters of the problem. Moreover, as has been discussed in [7], the field equation can be seen as the motion of particle in a two-dimensional flat space in which the motion is driven by the potential, $V_{eff}(a, R) = a^3(f'R - R)$. Hence for different potentials, functions $f(R)$, the system has a different evolution and that means that the scale factor, $a(t)$, evolves differently and this corresponds to different cosmological models. Therefore the functional form of $f(R)$ is essential.

Different models have been proposed in the literature, some of which can be found in [8–11] and references therein, while some models which describe the accelerated expansion of the universe can be found in [12]. However, there are only few known forms for the $f(R)$ function in which the field equations form an integrable dynamical system and consequently the solution can be written in a closed-form. Most of these forms have been found with the method of group invariant transformations [7, 13–15], while the results of [16, 17] on the integrable scalar-tensor theories can be used in order to construct solutions in $f(R)$ -gravity.

In order to perform the singularity analysis of the field equations, (5)-(7), we consider that the function, $f(R)$, has the functional form⁴ [9]

$$f(R) = R + qR^n, \quad (8)$$

where for $n = 2$ the well-known Starobinsky model follows [18]. Furthermore we assume that $q \neq 0$ and $n \neq 0, 1$. Otherwise the Action Integral (1) is that of GR. Hence we determine constraints on the power, n , in which the field equations, (6)-(7), pass the singularity test and the dynamical system is integrable in that the solution can be written always as a Laurent expansion. Specifically for (8) the Euler-Lagrange equations of (3) are equation (7) and

$$\ddot{R} + \frac{n-2}{R}\dot{R}^2 + \frac{2}{a}\dot{a}\dot{R} - \frac{(R + qnR^n)}{qn(n-1)}R^{1-n}a^{-2}\dot{a}^2 - \frac{(qR^{n-1}(n-3) - 2)}{qn(n-1)}R^{1-n} = 0, \quad (9)$$

² $H = \dot{a}/a$ is the Hubble Function.

³ The corresponding conservation law of the autonomous Noether symmetry, ∂_t , of the Lagrangian function (4).

⁴ For the cosmological implication of this model see [19].

while the constraint equation (5) becomes

$$3(1 + nqR^{n-1})a\dot{a}^2 + 3n(n-1)a^2R^{n-2}\dot{a}\dot{R} - q(n-1)a^3R^n = \rho_{m0} \quad (10)$$

The application of singularity analysis of differential equations in gravitational studies has been used for the study of the integrability of the field equations in GR for the Bianchi IX spacetime, the Mixmaster universe [21–23], the singularity analysis of the Mixmaster universe in a fourth-order gravity can be found in [23] while some other applications in [24].

For brevity we omit the basic properties of the singularity analysis which can be found for instance in [25–27] and references therein, in the following the ARS algorithm is applied [25]. Before we start the analysis for the model (8) with general n , we consider the well-known integrable case $n = \frac{4}{3}$, which has been found with the application of Killing tensors in the minisuperspace approach [15].

We start by searching for the dominant behaviour, $(a, R) \rightarrow (a_0\tau^p, R_0\tau^q)$, in the system (6)-(7) for $n = \frac{4}{3}$, where $\tau = (t - t_0)$. From (7), as it is expected, we find that $R_0 = 6(2p^2 - p)$, while from the second one we find from the dominant terms that

$$p_1 = 0, \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{5}{6}, \quad p_4 = \frac{8}{9} \quad (11)$$

for arbitrary a_0 . That is a physically expected result because a_0 can be always absorbed into the spacetime and does not provide any properties in the solution. The values, p_1 and p_2 , do not provide a singularity for the field equations and so we select to proceed with the value p_3 . While the latter is a fraction we can work as it is and we do not have to do any transformation $a \rightarrow a^\lambda$ in order to make the value of p negative.

The second step is to determine the resonances, s , which should be rational numbers, i.e., $s \in \mathbb{Q}$. To do that we substitute

$$a(\tau) = a_0\tau^{\frac{5}{6}} + \gamma\tau^{\frac{5}{6}+s}, \quad R(\tau) = \frac{10}{3}\tau^{-2} + \delta\tau^{\frac{5}{6}+s} \quad (12)$$

into (6)-(7) where γ, δ are two numbers which we want to be arbitrary and so give the additional constants of integration of the system. From this we find the following system

$$\left\{ \begin{array}{l} a_0\delta - 2s(7 + 3s)\gamma = 0 \\ a_0(25 - 8s + 12s^2)\delta - 360s\gamma = 0 \end{array} \right\} \quad (13)$$

which are required to have solution for the parameters, γ, δ , to be arbitrary. Hence the determinant of the system (13) has to be zero and this gives the resonances⁵

$$s_1 = -1, \quad s_2 = 0, \quad s_3 = \frac{1}{6}, \quad s_4 = -\frac{5}{6}. \quad (14)$$

Hence, as all the resonances are rational numbers, we say that the model (8) with $n = \frac{4}{3}$ passes the singularity test. Furthermore, because one of the resonances is zero, the coefficient of the leading-order term is arbitrary. As s_3, s_4 are smaller numbers than p_3 , the Laurent expansion is a mixed (Left and Right) Painlevé Series. Hence the solution of the scale factor is in the form

$$a(\tau) = \sum_{J=-2}^{-\infty} \alpha_J \tau^{\frac{5+J}{6}} + \alpha_{-1} \tau^{-\frac{4}{6}} + \alpha_0 \tau^{\frac{5}{6}} + \alpha_1 \tau^{\frac{4}{6}} + \sum_{I=2}^{+\infty} \alpha_I \tau^{\frac{5+I}{6}} \quad (15)$$

in which the coefficient constants have to be determined by substitution into the system (6)-(7). There is also the last test that one has to perform. This is the consistency of the solution when the nondominant terms are considered. However, for this case we know well that the system is integrable and it is not necessary to present the consistency test. Furthermore from (11) and for the dominator p_4 , we find the following resonances $s_1 = -1, s_2 = 0, s_3 = -\frac{2}{9}, s_4 = -\frac{7}{9}$, which give us the second solution

$$a(\tau) = \bar{\alpha}_0 \tau^{\frac{8}{9}} + \bar{\alpha}_1 \tau^{\frac{7}{9}} + \bar{\alpha}_2 \tau^{\frac{6}{9}} + \sum_{I=3}^{\infty} \bar{\alpha}_I \tau^{\frac{8-I}{9}}, \quad (16)$$

⁵ We note that expression (12) does not solve the field equations, that is, $R(\tau)$ in (12) is not given from (7) for the scale factor $a(\tau) = a_0\tau^{\frac{5}{6}} + \gamma\tau^{\frac{5}{6}+s}$. The analysis is equivalent either if we had applied it in the fourth-order equation (6), where R has been replaced from (7).

which is a Left Painlevé Series in contrary to the other solution which is expressed with a mixed Painlevé Series. The second solution is expected and from the results of [15]. However, before we proceed with the general case let us consider a second particular case of (8) in which $n = \frac{3}{2}$.

For the latter value of n the field equations do not admit any point symmetry or contact symmetry. That does not mean that a higher-order symmetry does not exist. In order to perform the singularity analysis we perform in (5)-(7) the change of variable $a(t) = (b(t))^{-1}$. Following the same steps we find the dominant behaviour $(b, R) \rightarrow (a_0\tau^p, R_0\tau^q)$, for $R_0 = 6(2p^2 + p)$ and

$$p_1 = 0, \quad p_2 = -\frac{1}{2}, \quad p_3 = -1, \quad p_4 = -2. \quad (17)$$

We consider the dominant exponent, $p_3 = -1$, for which the corresponding resonances are $s_1 = -1$, $s_2 = 0$, $s_3 = -2$, $s_4 = 1$, that is, the system passes the singularity test and it is integrable. The solution is given in a form of the Laurent expansion,

$$b(\tau) = \sum_{J=-3}^{-\infty} \beta_J \tau^{-1+J} + \beta_{-2} \tau^{-3} + \beta_{-1} \tau^{-2} + \beta_0 \tau^{-1} + \beta_1 + \beta_2 \tau + \sum_{I=3}^{+\infty} \beta_I \tau^{-1+I}, \quad (18)$$

and

$$R(\tau) = r_0 \tau^{-2} + r_1 \tau^{-1} + r_2 + \sum_{I=3}^{\infty} r_I \tau^{-2+I}. \quad (19)$$

In order to test the consistency of the solution (18), (19), we substitute into (5)-(7) where we derive the first coefficients β_I, r_I . Recall that, as one of the resonances is zero, β_0 has to be arbitrary. We find the following values $(r_0, r_1, r_2, \dots) = (6, -24\frac{\beta_1}{\beta_0}, 6(7\beta_1^2 - 10\beta_0\beta_2)\beta_0^{-2}, \dots)$, and $(\beta_2, \beta_3, \beta_4, \dots) = \frac{\beta_1}{\beta_0} \left(\frac{3\beta_1}{4}, \frac{\beta_1^2}{4\beta_0}, -\frac{13}{32}\frac{\beta_1^3}{\beta_0^2}, \dots \right)$. From this we observe that the constants β_0, β_1 are arbitrary, the third integration constant is hidden in the other coefficients while the fourth integration constant is the position of the singularity τ_0 . Furthermore from the exponent, $p_4 = -2$, we determine the second solution.

For the dominant behavior, $p_4 = -2$, the corresponding resonances are $s_1 = -1$, $s_2 = 0$, $s_3 = -3$ and $s_4 = -4$. Hence the solution is

$$b(\tau) = \bar{\beta}_0 \tau^{-2} + \beta_1 \tau^{-3} + \beta_2 \tau^{-4} + \sum_{I=3}^{\infty} \beta_I \tau^{-2-I} \quad (20)$$

and

$$R(\tau) = \bar{r}_0 \tau^{-2} + \bar{r}_1 \tau^{-3} + r_2 \tau^{-4} + \sum_{I=3}^{\infty} r_I \tau^{-2-I}, \quad (21)$$

which is a Left Painlevé Series. The first coefficients $\bar{\beta}_I$, and \bar{r}_I are: $(\bar{r}_0, \bar{r}_1, \bar{r}_2, \dots) = (36, 36\frac{\bar{\beta}_1}{\bar{\beta}_0}, -6(3\bar{\beta}_1^2 - 10\bar{\beta}_0\bar{\beta}_2)\bar{\beta}_0^{-2}, \dots)$ and $(\bar{\beta}_2, \bar{\beta}_3, \dots) = \frac{\bar{\beta}_1}{\bar{\beta}_0} \left(\frac{3\bar{\beta}_1}{4}, \frac{\bar{\beta}_1^2}{5\bar{\beta}_0}, \dots \right)$, which pass the consistency test. Finally if we substitute the solution in (5) then the relation among the integration constants and the matter density ρ_{m0} will be found.

We apply the same algorithm for general value n , and we find that there are only two possible leading terms p_I and p_{II} which depend on the value of n . We summarize the results in the following proposition.

Proposition: *The gravitational field equations, (5)-(7), of the fourth-order theory of gravity $R + qR^n$, with $q \neq 0$, $n \neq 0, 1$, pass the singularity test for $n > 1$, $n \neq 2, \frac{5}{4}$, in which the following set of leading-order exponents and of the resonances are rational numbers.*

I) For the dominant term $p_I = \frac{2n^2 - 3n + 1}{n - 2}$ with $p_I \in \{\mathbb{Q} - \mathbb{N}\}$ the corresponding resonances are

$$s_1 = -1, \quad s_2 = 0, \quad s_3 = \frac{5 - 14n + 8n^2}{n - 2}, \quad s_4 = 1 + s_3. \quad (22)$$

while for $p_I \in \mathbb{N}$ we make the change of variable $a(\tau) \rightarrow b^{-1}(\tau)$ in which the new dominant term is $\bar{p}_I = -p_I$ and the resonances are the same.

II) For the dominant term $p_{II} = \frac{2}{3}n$, in which $p_{II} \in \{\mathbb{Q} - \mathbb{N}\}$ the corresponding resonances are

$$s_1 = -1, \quad s_2 = 0, \quad s_{\pm} = \frac{3(1 - n) \pm \sqrt{(n - 1)(256n^3 - 608n^2 + 417n - 81)}}{6(n - 1)}. \quad (23)$$

while, when $p_{II} \in \mathbb{N}$, we make the change of variable $a(\tau) \rightarrow b^{-1}(\tau)$, in which the new dominant term is $\bar{p}_{II} = -p_{II}$ and the resonances are the same.

It is important to mention that from (23), when n is a rational number, then the dynamical system is always integrable with leading term p_I . The latter of course is not true for the exponents p_{II} .

Furthermore the solution of the scale factor in the singularity is described from the dominant term, that is, from the power-law solution $a(t) \simeq \tau^p$, which corresponds to a cosmological solution in which the total fluid has a constant equation of state parameter

$$w_{tot} = \frac{p}{\rho} = -\frac{6n^2 - 11n + 7}{3(2n-1)(n-1)}. \quad (24)$$

For the latter for the range of n in which the field equations (5)-(7) pass the singularity analysis we have that $w_{tot} < 1$. Specifically $w_{tot} < -1$, for $n \in (1, 2)$ and $w_{tot} > -1$ for $n > 2$, while for large values of n , we have that $\lim_{n \rightarrow \infty} w_{tot} = -1^+$.

However, expression (24) is that of the $f(R) = R^n$ gravity has been found [28], which means that, when the system (5)-(7) reaches the singularity, then the model (8) has the behaviour⁶ of R^n .

The consistency test have been applied for two special values of n , the value $n_1 = \frac{4}{3}$ and $n_2 = \frac{3}{2}$, which both corresponds to the two possible cases of the above proposition. To perform the consistency test for general n is difficult since the step on the Laurent expansion will be unknown as also the position of the integration constants. These depend on the value of n .

We introduced the singularity analysis of differential equations to study the integrability of higher-order theories of gravity. Specifically we applied the method on $f(R)$ -gravity in the metric formalism. We considered a special form of $f(R)$ which has been proposed in [9] and we found that for specific values of n , with $n > 1$, the gravitational field equations in a spatially FLRW spacetime are integrable. The singularity analysis is complementary to the symmetry analysis. Symmetries, as we discussed above, have been used for the determination of the unknown potentials/models in different gravitational theories and consequently for the determination of new integrable systems in cosmological studies. However, only the point symmetries and the contact symmetries have been applied till now and that does not mean that there are not systems which admit other kind of symmetries, such as nonlocal symmetries or Lie-Bäcklund symmetries which in general are difficult to calculate.

Furthermore for the case (I) of the above proposition we would like to give the ranges of n , in which the analytical solution is expressed in a Right or Left Painlevé Series. For values of n in which p_I is not a positive integer number, for $n \in (1, \frac{5}{4}) \cup (2, n)$ the solution is expressed from a Right Painlevé Series and for $n \in (\frac{13}{6} + \sqrt{\frac{73}{6}}, 2)$ the solution is expressed in a Left Painlevé Series, while for $n \in (\frac{5}{4}, \frac{13}{6} + \sqrt{\frac{73}{6}})$ the Painlevé Series has left and right expansions.

As we discussed above $f(R)$ gravity is equivalent with O'Hanlon gravity, that is, the Action integral (1) can be written as that of Brans-Dicke Action, with Brans-Dick parameter zero, $\omega_{BD} = 0$, where the latter is equivalent with a minimally coupled scalar field cosmological model under a conformal transformation.

Consider that $\phi = f_{,R}$, then for (8) we find, $R^{n-1} = (\frac{1}{nq})(\phi - 1)$, where in the case of vacuum Lagrangian (3) becomes,

$$L(N, a, \dot{a}, R, \dot{R}) = \frac{1}{N} (6a\phi\dot{a}^2 + 6a^2\dot{a}\dot{\phi}) + N (a^3 V_0 (\phi - 1)^{\frac{n}{n-1}}), \quad (25)$$

in which $V_0 = q(n-1)(nq)^{-\frac{1}{n-1}}$. Consider $a(t) = A(t)\phi(t)^{-\frac{1}{2}}$, and $N(t) = \bar{N}(t)\phi(t)^{-\frac{1}{2}}$, hence Lagrangian (25) becomes

$$L(\bar{N}, A, \dot{A}, \psi, \dot{\psi}) = \frac{1}{\bar{N}} \left(3A\dot{A}^2 - \frac{1}{2}A^3\dot{\psi}^2 \right) + \bar{N}A^3\bar{V}_0 \left(e^{-\left(\frac{4}{3\mu} + \frac{2}{3}\right)\psi} - e^{-\frac{4}{3\mu}\psi} \right)^\mu \quad (26)$$

where $\mu = \frac{n}{n-1}$, $V_0 = 2\bar{V}_0$ and the new field $\psi(t)$ is related with $\phi(t)$ with the formula $\ln \phi = \frac{2}{3}\psi$. Therefore from (26) we have that the potential of the minimally coupled scalar field is

$$V(\psi) = \bar{V}_0 \left(e^{-\left(\frac{4}{3\mu} + \frac{2}{3}\right)\psi} - e^{-\frac{4}{3\mu}\psi} \right)^\mu \quad (27)$$

which is not included in the list of integrable potentials of [16].

In a forthcoming work we wish to extend our analysis to other proposed models which can be found in the literature and also to study the physical properties of these models which are found to be integrable.

⁶ The dynamics of that model can be found in [29]

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