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Collapsing radiating stars with various equations of state

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We study the gravitational collapse of radiating stars in the context of the cosmic censorship conjecture. We consider a generalized Vaidya spacetime with three concentric regions. The local internal atmosphere is a two-component system consisting of standard pressure-free, null radiation and an additional string fluid with energy density and nonzero pressure obeying all physically realistic energy conditions. The middle region is purely radiative which matches to a third region which is the Schwarzschild exterior. We outline the general mathematical framework to study the conditions on the mass function so that future-directed nonspacelike geodesics can terminate at the singularity in the past. Mass functions for several equations of state are analyzed using this framework and it is shown that the collapse in each case terminates at a locally naked central singularity. We calculate the strength of these singularities to show that they are strong curvature singularities which implies that no extension of spacetime through them is possible.

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I. INTRODUCTION

The Vaidya metric [1] describes the geometry outside a spherically symmetric radiating star and it defines outgoing null radiation. When a supermassive star of mass greater than eight solar masses reaches the end of the luminous phase of its life, it experiences an inwardly directed gravitational collapse. This is a very violent process which occurs on time scales of the order of seconds and is observed as a type II supernova. The entire collapse process is usually divided into early, intermediate, and late stages. The effects of radiation are important in the later stages of gravitational contraction when an immense amount of energy is ejected from the star in the form of neutrinos or photons.

The notion of collapse was first brought to light by Oppenheimer and Snyder [2], and they described the free-fall contraction of a spherical body in which pressure forces were completely overwhelmed by the gravitational forces. The equations of collapse—as analyzed analytically in Refs. [3–7] and numerically in Refs. [8–11]—have produced significant new insights into gravitational collapse. A supermassive stellar object, in its very long life, will exist in a state of suspended collapse, converting its hydrogen into helium, carbon, neon, oxygen, magnesium, and silicon through nucleosynthesis and creating an internal pressure gradient resulting in the release of outward energy (radiation, convection, and conduction). Thermonuclear fusion ends at iron-56, the most bound nuclear species. Beyond iron, fusion is no longer exothermic.

The process of gravitational collapse is complicated. Concentric burning shells are created as one element after the other is synthesized. Enormous amounts of gamma rays in the core produce electron-positron pairs which annihilate, producing neutrino pairs. Iron-56 is the end point of nucleosynthesis: a hydrodynamical instability sets in and at this point gravitational forces will crush the core to an extent where the electrons become relativistic. The infalling material in the core overshoots the equilibrium configuration and rebounds from the stiffened core. This generates a post-bounce-presupernova shockwave which propagates outward from some point within the collapsing core reaching relativistic speeds. It is understood that it is this shockwave that drives the outflow of neutrinos. In view of this, Glass [12] modeled the emission of neutrinos in dissipative collapse, and Herrera and Núñez [13] investigated the associated shock structure and propagation in the interior of the radiating star. As this shockwave travels outward, its energy is dissipated by neutrino losses and by photodisintegration of all the nuclei in its path, and eventually stalls. The infalling plasma material in the star (which surrounded the precollapsed core) fills the available space, generating a decompression shockwave which travels radially outward at the speed of sound through this diffuse stellar material (which now begins to free fall). The free-falling material is seized as it meets the shock front, and this turns the latter into an accretion shock which is heated by this falling plasma. A rarefied bubble-like region develops between the collapsing core and the accreting shock front and neutrino pairs diffusing from the extremely hot interior annihilate, expanding this bubble. Through a complex series of events this shock front drives out all the matter in a type II supernova. It is understood that the final result of collapse is a black hole (or collapsar).

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A. Generalized Vaidya spacetime

The generalization of the Vaidya spacetime was given in detail by Wang and Wu [14] and includes most of the known solutions of Einstein's field equations with the additional type II fluid. These were further extended by Brassel *et al.* [15] for various equations of state. It is written in terms of the mass of the radiating body and the Petrov-Pirani-Penrose classification of the metric is type D [16]. The notion that the energy-momentum tensor is linear in terms of the gravitational mass for these matter fields prompts this generalization of the spacetime. The generalized Vaidya metric in single (exploding) null coordinates $(v, \mathsf{r}, \theta, \phi)$ is given as

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right)dv^2 - 2dvd\mathsf{r} + \mathsf{r}^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

Here the function $m(v, \mathsf{r})$ describes the Misner-Sharp mass of the stellar interior and can be obtained via integrating the Einstein field equations with combinations of perfect fluid and null matter sources.

Generalized Vaidya spacetimes have been widely used in the study of regular and dynamical black holes [17,18] as well as black holes with trapped regions [19]. The Vaidya-Papapetrou model [20,21] is one of the earliest to counter the cosmic censorship conjecture (CCC). Here, a physically reasonable matter field satisfying the energy conditions was found in a shell-focusing central singularity ($v = 0$, $\mathsf{r} = 0$) which was formed by imploding shells of radiation. Radially injected radiation flows into a region which is initially flat, and is focused into a central singularity of increasing mass. A central singularity was shown to become a node with a definite tangent for families of nonspacelike geodesics, for some nonvanishing measure of parameters in the model. Thus the singularity is naked, as families of future-directed nonspacelike geodesic curves going to future null infinity terminate at the central singularity in the past. A comprehensive analysis on censorship violation was given in Refs. [22,23]. Mkenyeleye *et al.* [24] studied the gravitational collapse of Vaidya spacetimes in the context of the CCC. A general mathematical framework was developed to study the conditions on the mass function where future-directed nonspacelike geodesics can terminate at the central singularity in the past. We will use this framework extensively in this paper. The results obtained were further generalized in higher dimensions in Ref. [25]. Maharaj *et al.* [26] showed that the generalized Vaidya model can be matched smoothly to a heat-conducting interior in the Santos framework [27]. The physical behavior of this model was analyzed in detail by the authors of Ref. [28] who investigated the effect of the exterior energy density on the temporal evolution of the radiating fluid pressure, luminosity, gravitational redshift, mass flow, and collapse rate at the boundary of a relativistic star.

B. This paper

The main intent of this paper is to study the gravitational collapse of generalized Vaidya spacetimes in the context of the CCC. The mass functions obtained in Ref. [15] for various equations of state will be analyzed using the general framework developed in Ref. [24]. It will be shown that for each mass function, the collapse terminates with a local central singularity, which is naked. We also calculate the strength of the naked singularities and show that they are strong curvature singularities and there does not exist an extension of spacetime through these singularities. This paper is organized as follows. In Sec. II we present a complete outline of how to model an isolated spherical and physically reasonable radiating astrophysical star via the generalized Vaidya geometry. In Sec. III we describe the generalized Vaidya spacetime by analyzing the field equations; the relevant aphorisms indicative of the geometry of the generalized Vaidya metric are presented and we mention the energy conditions for a physically reasonable model. In Sec. IV we systematically present the mathematical framework of Ref. [24] for a collapsing model, and in Sec. V we present the conditions for the formation of a locally naked singularity and its strength. Section VI details the end state of generalized mass functions found by Brassel *et al.* [15] for various equations of state.

II. THE MODEL OF A RADIATING AND DYNAMIC RELATIVISTIC STAR

A spherically symmetric isolated astrophysical star is a combination of three distinct and concentric zones. The innermost zone is the stellar interior where there is null fluid matter along with radiation. The middle zone is purely a radiation zone. The outermost zone is the vacuum Schwarzschild exterior that extends to a radius of roughly 1 light year (for solar mass stars), beyond which galactic dynamics begin to take over. In this section we briefly outline how to model all three of these zones under a combined framework using a generalized Vaidya class of metric. The full details of these notions can be found in Ref. [15].

A. Matching conditions at the boundary layers: Complete mass function

We note here that the spacetime is divided into three distinct regions: the stellar interior, the radiation zone, and the Schwarzschild exterior region. The first boundary layer between the inner and the intermediate zone, given by $\mathsf{r} = \mathsf{r}_b$, is a timelike boundary, whereas the second boundary given by $v = V_0$ is a null boundary. The important point that all three zones are described by the same class of metric makes the matching conditions across these boundaries extremely transparent.

To match the first fundamental form all we need is the mass function to be continuous across these boundaries.

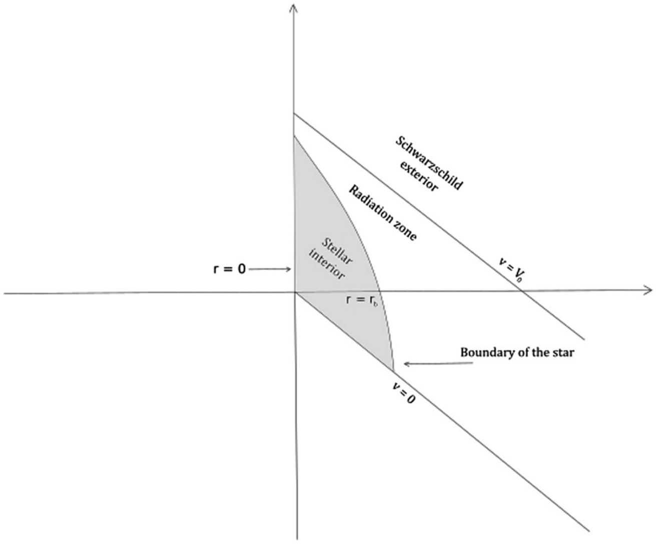

FIG. 1. Depiction of spacetime divided into the three distinct regions: the stellar interior, a purely radiative zone, and the vacuum exterior.

Hence the complete C^2 mass function for an isolated stellar model can be given in the following form:

$$m(v,\mathsf{r}) = \begin{cases} m(v,\mathsf{r}) & \mathsf{r} \le \mathsf{r}_b,\, v \le V_0, \\ m_1(v) \equiv m(v,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b,\, v \le V_0, \\ M \equiv m_1(V_0) \equiv m(V_0,\mathsf{r}_b) & \mathsf{r} > \mathsf{r}_b,\, v > V_0. \end{cases} \tag{2}$$

We can easily check that this mass function is a solution to Einstein's field equations in all three zones mentioned above, and hence it completely describes the spacetime of an isolated star that is collapsing. An important point to note is the boundary $v = V_0$. Between the intermediate and exterior zones, matter must be infalling in order for the exterior to be vacuum. Also, as will be discussed later, matter which is directed inward falls to a central point which is, in essence, a singularity. To match the second fundamental form, we need the partial derivatives of the mass functions across the boundaries to be continuous. These conditions are given by

$$\frac{\partial}{\partial v} m(v,\mathsf{r}_b) = \frac{\partial}{\partial v} m_1(v), \tag{3a}$$

$$\frac{\partial}{\partial \mathsf{r}} m(v,\mathsf{r})|_{\mathsf{r}=\mathsf{r}_b} = 0, \tag{3b}$$

$$\frac{\partial}{\partial v} m_1(v)|_{v=V_0} = 0, \tag{3c}$$

where $\mathsf{r} = \mathsf{r}_b$ is the timelike boundary and $v = V_0$ is the null boundary. These boundaries serve as the matching surfaces for the three concentric regions, which can be seen in Fig. 1.

So the complete picture is as follows. We have a spherically symmetric distribution of static or dynamic *type I* matter (a perfect fluid), through which a thick shell of collapsing null radiation is superposed. This system finally collapses to a spacetime singularity. Here $v = 0$ depicts the first null ray that falls into the central singularity formed at $\mathsf{r} = 0$. In this paper we will rigorously analyze the nature of this singularity in terms of its visibility to faraway observers. Of course in this analysis we are using only collapsing null shells. On top of this we can also superimpose outgoing shells from the nontrapped regions, that go to infinity. However, this will not effect the calculations around the central singular point which lies in the boundary of the trapped region in the interior spacetime.

III. FIELD EQUATIONS AND ENERGY CONDITIONS

The line element for all three regions belongs to the generalized Vaidya class given by Eq. (1). Note that $m(v,\mathsf{r})$ is the mass of the star and is related to the gravitational energy within a given radius r [29,30]. From the above we have the following quantities:

$$R^0{}_0 = R^1{}_1 = \frac{m_{\mathsf{rr}}}{\mathsf{r}}, \tag{4a}$$

$$R^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{4b}$$

$$R^2{}_2 = R^3{}_3 = \frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{4c}$$

with the Ricci scalar

$$R = \frac{2}{\mathsf{r}^2}(\mathsf{r}m_{\mathsf{rr}} + 2m_{\mathsf{r}}),$$

where we have used the notation

$$m_v = \frac{\partial m}{\partial v}, \qquad m_{\mathsf{r}} = \frac{\partial m}{\partial \mathsf{r}}.$$

The Einstein tensor components are

$$G^0{}_0 = G^1{}_1 = -\frac{2m_{\mathsf{r}}}{\mathsf{r}^2}, \tag{5a}$$

$$G^1{}_0 = \frac{2m_v}{\mathsf{r}^2}, \tag{5b}$$

$$G^2{}_2 = G^3{}_3 = -\frac{m_{\mathsf{rr}}}{\mathsf{r}}. \tag{5c}$$

The energy-momentum tensor is defined by

$$T_{ab} = T^{(n)}_{ab} + T^{(m)}_{ab}, \tag{6}$$

where

$$T^{(n)}_{ab} = \mu l_a l_b,$$
$$T^{(m)}_{ab} = (\tilde{\rho} + P)(l_a n_b + l_b n_a) + P g_{ab}.$$

In the above,

$$l_a = \delta^0_a, \qquad n_a = \frac{1}{2}\left[1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right]\delta^0_a + \delta^1_a,$$

with $l_c l^c = n^c_c = 0$ and $l_c n^c = -1$. The null vector l^a is a double null eigenvector of the energy-momentum tensor (6). Hence, the nonzero components are given by

$$T^0{}_0 = -\tilde{\rho}, \tag{7a}$$

$$T^1{}_0 = -\mu, \tag{7b}$$

$$T^2{}_2 = T^3{}_3 = P. \tag{7c}$$

The Einstein field equations ($G^a{}_b = \kappa T^a{}_b$) become

$$\mu = -2\frac{m_v}{\kappa \mathsf{r}^2}, \tag{8a}$$

$$\tilde{\rho} = 2\frac{m_{\mathsf{r}}}{\kappa \mathsf{r}^2}, \tag{8b}$$

$$P = -\frac{m_{\mathsf{rr}}}{\kappa \mathsf{r}}, \tag{8c}$$

which describe the gravitational behavior of a string fluid [31,32].

The energy conditions for this kind of fluid are given by the following.

(1) The weak and strong energy conditions:

$$\mu \geq 0, \qquad \tilde{\rho} \geq 0, \qquad P \geq 0 \qquad (\mu \neq 0). \tag{9}$$

(2) The dominant energy condition:

$$\mu \geq 0, \qquad \tilde{\rho} \geq P \geq 0 \qquad (\mu \neq 0). \tag{10}$$

In the case when $m = m(v)$ the above energy conditions all reduce to $\mu \geq 0$, and if $m = m(r)$, then $\mu = 0$ and the matter field becomes a type I fluid.

IV. COLLAPSING MODEL

We will examine the gravitational contraction of imploding matter and radiation described by the generalized Vaidya spacetime. Here, a thick shell of radiation and type I matter collapses at the center of symmetry [22]. If K^a is the tangent to nonspacelike geodesics with $K^a = \frac{dx^a}{d\hat{k}}$, where $\hat{k}$ is the affine parameter, then $K^a{}_{;b}K^b = 0$ and

$$g_{ab}K^a K^b = \mathcal{B}, \tag{11}$$

where $\mathcal{B}$ is a constant which characterizes different classes of geodesics. Null geodesics are characterized by $\mathcal{B} = 0$, while $\mathcal{B} < 0$ applies to timelike geodesics. The equations for the quantities $\frac{dK^v}{d\hat{k}}$ and $\frac{dK^{\mathsf{r}}}{d\hat{k}}$ are calculated from the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\varrho} - \frac{d}{d\hat{k}}\left(\frac{\partial L}{\partial \dot{x}^\varrho}\right) = 0, \tag{12}$$

where the Lagrangian is

$$L = \frac{1}{2} g_{ab}\dot{x}^a \dot{x}^b. \tag{13}$$

In the above equations the dot denotes differentiation with respect to the affine parameter $\hat{k}$. These equations were given in Ref. [24]. The components K^θ and K^ϕ of the tangent vector are

$$K^\theta = \frac{l\cos\varphi}{\mathsf{r}^2\sin^2\theta}, \tag{14a}$$

$$K^\phi = \frac{l\sin\varphi\cos\phi}{\mathsf{r}}, \tag{14b}$$

where l is the impact parameter and φ is the isotropy parameter defined by $\sin\phi\tan\varphi = \cot\theta$.

Following Ref. [21], we can write K^v as

$$K^v = \frac{P}{\mathsf{r}}, \tag{15}$$

where $P = P(v, \mathsf{r})$ is some arbitrary function. Therefore, $g_{ab}K^a K^b = \mathcal{B}$ gives

$$K^{\mathsf{r}} = \frac{P}{2\mathsf{r}}\left[1 - \frac{2m(v,\mathsf{r})}{\mathsf{r}}\right] - \frac{l^2}{2\mathsf{r}P} + \frac{\mathcal{B}\mathsf{r}}{2P}. \tag{16}$$

Using Eq. (15), we calculate $\frac{dK^v}{d\hat{k}}$ and thus

$$\frac{dP}{d\hat{k}} = \frac{1}{\mathsf{r}}\left(\mathsf{r}^2\frac{dK^v}{d\hat{k}} + P\frac{d\mathsf{r}}{d\hat{k}}\right). \tag{17}$$

If we substitute the Euler-Lagrange equations and Eq. (16) into Eq. (17), the differential equation satisfied by the function P results in

$$\frac{dP}{d\hat{k}} = \frac{P^2}{2\mathsf{r}^2}\left(1 - \frac{4m(v,\mathsf{r})}{\mathsf{r}} + 2m'(v,\mathsf{r})\right) + \frac{l^2}{2\mathsf{r}^2} + \frac{\mathcal{B}}{2}. \tag{18}$$

If the mass function $m(v,\mathsf{r})$ and initial conditions are defined, the function $P(v,\mathsf{r})$ can then be found.

V. THE CONDITIONS FOR A LOCALLY NAKED SINGULARITY

In this section we analyze how the final fate of collapse is determined in terms of a naked singularity or a black hole, given the generalized Vaidya mass function. If there exist families of future-directed nonspacelike trajectories reaching observers far away in spacetime, which terminate in the past at the singularity, then the singularity forming as the final state of collapse is naked. If no such families exist and an event horizon forms at a sufficiently early time to cover the singularity, we then have a black hole. The equation of null geodesics for the metric (1) is given by

$$\frac{dv}{d\mathsf{r}} = \frac{2\mathsf{r}}{\mathsf{r} - 2m(v,\mathsf{r})}. \tag{19}$$

This equation has a singularity at $v = 0$ and $\mathsf{r} = 0$ and its nature can be analyzed using the standard techniques associated with the theory of differential equations [33–35].

A. Structure of the central singularity

Equation (19) can generally be written in the separable form

$$\frac{dv}{d\mathsf{r}} = \frac{\hat{M}(v,\mathsf{r})}{\hat{N}(v,\mathsf{r})}, \tag{20}$$

with its singularity at $v = \mathsf{r} = 0$, where the functions $\hat{M}$ and $\hat{N}$ are vanquished. Thus, the analysis of the existence and uniqueness of the solution to this differential equation should be carefully considered. If we introduce the new independent variable t with differential dt such that

$$\frac{dv}{\hat{M}(v,\mathsf{r})} = \frac{d\mathsf{r}}{\hat{N}(v,\mathsf{r})} = dt, \tag{21}$$

the differential equation (20) may be replaced by

$$\frac{dv(t)}{dt} = \hat{M}(v,\mathsf{r}), \tag{22a}$$

$$\frac{d\mathsf{r}(t)}{dt} = \hat{N}(v,\mathsf{r}). \tag{22b}$$

It can be easily seen that the singular point of Eq. (20) is a fixed point of the system (22). In order to find the necessary and sufficient conditions for the existence of solutions to this system in the region of the fixed point $v = \mathsf{r} = 0$, Eq. (22) can be written as a differential equation of the vector $\mathbf{x}(t) = [v(t), \mathsf{r}(t)]^T$ on $\Re^2$ as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)). \tag{23}$$

Several definitions and theorems were given in Ref. [24] on the methodologies of showing the existence and uniqueness of solutions to the above system (22) and Eq. (23).

B. Nature of the fixed point $v = \mathsf{r} = 0$

Since the partial derivatives of the functions $\hat{M}$ and $\hat{N}$ exist and are continuous in the neighborhood of the fixed point, the system can be linearized near the fixed point and thus the general behavior of this system near the singularity is homologous to the characteristic equations

$$\frac{dv}{dt} = Av + B\mathsf{r}, \tag{24a}$$

$$\frac{d\mathsf{r}}{dt} = Cv + D\mathsf{r}, \tag{24b}$$

where $A = \hat{M}_v(0,0)$, $B = \hat{M}_\mathsf{r}(0,0)$, $C = \hat{N}_v(0,0)$, and $D = \hat{N}_r(0,0)$, with

$$\hat{M}_v = \frac{\partial \hat{M}}{\partial v}, \qquad \hat{M}_\mathsf{r} = \frac{\partial \hat{M}}{\partial \mathsf{r}},$$

and

$$\hat{N}_v = \frac{\partial \hat{N}}{\partial v}, \qquad \hat{N}_\mathsf{r} = \frac{\partial \hat{N}}{\partial \mathsf{r}}.$$

In the above, $AD - BC \neq 0$. The singularity of Eq. (24) can be classified as a node if $(A-D)^2 + 4BC \geq 0$ and $BC > 0$. It is otherwise a center of focus. In Eq. (19) we have that $M(v,\mathsf{r}) = 2\mathsf{r}$ and $N(v,\mathsf{r}) = \mathsf{r} - 2m(v,\mathsf{r})$. If $v = 0$ and $\mathsf{r} = 0$ at the central singularity we can define the following limits:

$$m_0 = \lim_{\substack{v\to 0\\ \mathsf{r}\to 0}} m(v,\mathsf{r}), \tag{25a}$$

$$\dot{m}_0 = \lim_{\substack{v\to 0\\ \mathsf{r}\to 0}} \frac{\partial}{\partial v} m(v,\mathsf{r}), \tag{25b}$$

$$m_0' = \lim_{\substack{v\to 0\\ \mathsf{r}\to 0}} \frac{\partial}{\partial \mathsf{r}} m(v,\mathsf{r}). \tag{25c}$$

The null geodesic equation can then be linearized near the central singularity as

$$\frac{dv}{d\mathsf{r}} = \frac{2\mathsf{r}}{(1-2m_0')\mathsf{r} - 2\dot{m}_0 v}. \tag{26}$$

It can be clearly seen that this equation has a singularity at $v = 0$ and r. It is possible to determine the nature of this singularity by observing the discriminant value of the characteristic equation. The roots of the characteristic equation are

$$\chi = \frac{1}{2}\left((1-2m_0') \pm \sqrt{(1-2m_0')^2 - 16\dot{m}_0}\right). \tag{27}$$

In order for the singularity at $v = 0$ and $\mathsf{r} = 0$ to be a node, we must have that

$$(1 - 2m'_0)^2 - 16\dot{m}_0 \geq 0, \qquad \dot{m}_0 > 0. \tag{28}$$

Hence, if the mass function $m(v, \mathsf{r})$ is chosen such that the above condition (28) is satisfied, the singularity at the origin will then be a node and outgoing nonspacelike geodesics can exit the singularity with a defined value of the tangent.

C. Existence of outgoing nonspacelike geodesics

We can now choose a generalized Vaidya mass function with the following properties.

(1) The mass function $m(v, \mathsf{r})$ obeys all of the physically reasonable energy conditions throughout the spacetime.
(2) The partial derivatives of the mass function must exist and are continuous on the entire spacetime.
(3) The limits of the partial derivatives of the mass function $m(v, \mathsf{r})$ at the central singularity obey the conditions $(1 - 2m'_0)^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$.

A mass function with the above properties would ensure the existence and uniqueness of the solutions of the null geodesic equation in the immediate vicinity of the central singularity. Also, the central singularity will be a node of C^1 solutions with definite tangents.

If we consider the tangents of these curves at the singularity, we can find the condition for the existence of outgoing radial nonspacelike geodesics from the nodal singularity. Let X be the tangent to the radial null geodesic. If the limiting value of X is finite and positive at the singular point, we can then see that the outgoing future-directed null geodesics terminate in the past at the central singularity. The existence of these radial null geodesics characterizes the nature (a naked singularity or a black hole) of the collapsing solutions. To determine the nature of the limiting value of X at $v = 0$ and $\mathsf{r} = 0$, we define the following:

$$X_0 = \lim_{\substack{v\to 0\\ \mathsf{r}\to 0}} X = \lim_{\substack{v\to 0\\ \mathsf{r}\to 0}} \frac{v}{\mathsf{r}}. \tag{29}$$

Using Eq. (26) and l'Hôpital's rule (for the C^1 null geodesics) and simplifying, we acquire

$$X_0 = \frac{2}{(1 - 2m'_0) - 2\dot{m}_0 X_0}. \tag{30}$$

Solving the above for X_0 gives the following:

$$X_0 = b_\pm = \frac{(1 - 2m'_0) \pm \sqrt{(1 - 2m'_0)^2 - 16\dot{m}_0}}{4\dot{m}_0}. \tag{31}$$

If one or more positive real roots exist for Eq. (30), then the singularity may be locally naked if the null geodesic lies outside the trapped region. In the following subsection we will consider the dynamics of the trapped region to find conditions for the existence of such geodesics.

D. Apparent horizon

The causal behavior of the trapped surfaces developing within the spacetime usually decides the occurrence of either a naked singularity or a black hole during the collapse evolution. The apparent horizon is the boundary of the trapped surface region within the spacetime. The equation of the apparent horizon for the generalized Vaidya spacetime is given as

$$\frac{2m(v, \mathsf{r})}{\mathsf{r}} = 1. \tag{32}$$

Hence, the slope of the apparent horizon can be calculated at the central singularity ($v \to 0$, $\mathsf{r} \to 0$) as

$$\left(\frac{dv}{d\mathsf{r}}\right)_{AH} = \frac{1 - 2m'_0}{2\dot{m}_0}. \tag{33}$$

All of the above can now be stated in the following proposition.

Proposition 1. Consider a collapsing generalized Vaidya spacetime from some regular epoch, with a mass function $m(v, \mathsf{r})$ that obeys all of the physically reasonable energy conditions and is differentiable in the entire spacetime. The central singularity is locally naked with outgoing C^1 radial null geodesics escaping to the future if the following conditions are satisfied:

(i) The limits of the partial derivatives of the mass function $m(v, \mathsf{r})$ at the central singularity obey the conditions $(1 - 2m'_0)^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$.
(2) There exists at least one root X_0 (real and positive) of Eq. (30).
(3) At least one of the positive real roots is less than $(\frac{dv}{d\mathsf{r}})_{AH}$ at the central singularity.

VI. STRENGTH OF THE SINGULARITY

If we consider the null geodesics parametrized by the affine parameter $\hat{k}$ and terminating at the shell-focusing singularity $v = \mathsf{r} = \hat{k} = 0$, we can compute the strength of the singularity (according to Ref. [36]). The strength of the singularity is the measure of its destructive capacity in the sense of whether the extension of spacetime is possible through the singularity or not [37]. Following Clarke and Krolack [38] and Mkenyeleye [24], a singularity would be strong if the condition

$$\lim_{\hat{k}\to 0} \hat{k}^2 \psi = \lim_{\hat{k}\to 0} \hat{k}^2 R_{ab} K^a K^b > 0 \tag{34}$$

is satisfied, where R_{ab} is the Ricci tensor. Given a suitable mass function, from Ref. [24] it can be shown that

$$\lim_{\hat{k}\to 0} \hat{k}^2 \psi = \frac{1}{4} X_0^2 (2\dot{m}_0) > 0. \tag{35}$$

If this above condition is indeed satisfied for some positive and real root X_0, then we can conclude that the naked singularity observed is strong. An interesting note is that when the energy conditions are satisfied, and if a naked singularity is developed as an end state of collapse, the naked singularity is always strong.

VII. END STATE OF GENERALIZED VAIDYA SPACETIMES FOR SEVERAL EQUATIONS OF STATE

In this section we present solutions to the Einstein field equations (8) for various equations of state, as found by Brassel *et al.* [15]. A direct integration of the resulting partial differential equations was possible in general for the linear, quadratic, and polytropic equations. Those solutions for the linear cases generalize all of those obtained by Husain [39] and others, as well as the complete summary of solutions presented in Ref. [14], and are therefore the most general solutions known. Also, using Eq. (30), the equations of the tangents to the null geodesics at the central singularity are calculated for these various Vaidya mass functions. We show that it is possible to obtain at least one real and positive value of X_0 for each mass function, and each of these mass functions are open sets in their functional space. Below, we include each solution with its equation of state.

(1) For the linear equation of state $P = k\tilde{\rho}$ the mass function found is

$$m(v, \mathsf{r}) = c_1(v)\frac{\mathsf{r}^{1-2k}}{1-2k} + c_2(v). \quad (36)$$

Using Eq. (30), we obtain the following:

$$2X_0^{2k+1} + 2\dot{c}_2 X_0^2 - X_0 + 2 = 0. \quad (37)$$

When $\dot{c}_2 = 0.01$ and $k = -1$, the above equation becomes a cubic and it is possible to find two positive and real roots. One such root is $X_0 = 2.862560272$ which means the singularity is naked. We also have that $c_2 = 0.01v$ and $c_1 = v^{2k}$ [from Eq. (25)] for $v > 0$. In Fig. 2, a naked singularity forms at the origin and we have a static distribution of matter focused into this central singularity of growing mass. Null radiation shells fall through this static distribution terminating at the singularity. For $v > 0$ and those values of c_1 and c_2 we have infalling light-like matter described by the generalized Vaidya metric [specifically, Eq. (36)] reaching the singularity. We also have that $\lim_{\hat{k}\to 0}\hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) = 0.04097 > 0$ so the condition for a strong singularity is satisfied. The real root is less than the slope at the apparent horizon,

$$X_0 < \left(\frac{dv}{d\mathsf{r}}\right)_{AH} = 37.79632254,$$

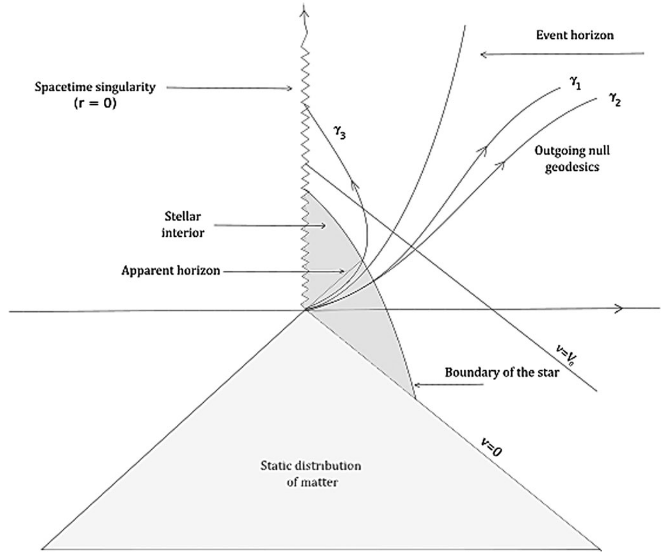

FIG. 2. Linear: Here $v = 0$ depicts the first collapsing null ray falling into the central singularity through a static distribution of matter. A naked singularity forms at the origin with families of trajectories γ_1 and γ_2 escaping to infinity from the singularity. A shell of null radiation falls through a static distribution of matter, and into the singularity. Nonspacelike curves such as γ_3, which are emitted after the event horizon, cross the apparent horizon and fall back into the singularity.

and thus the third condition of *Proposition 1* is satisfied. Therefore, the central singularity is naked and strong with outgoing C^1 radial null geodesics escaping to the future.

(2) For the generalized linear equation of state $P = k\tilde{\rho} + k_2$ we have

$$m(v, \mathsf{r}) = \frac{-\kappa k_2}{3(2k+2)}\mathsf{r}^3 + \frac{c_1 \mathsf{r}^{1-2k}}{1-2k} + c_2. \quad (38)$$

By using Eq. (30), the equation we obtain is identical to Eq. (37) and so all of the conditions of *Proposition 1* are satisfied as well. An interesting observation is that this solution generalizes all those contained in Refs. [14,39], so those particular cases should satisfy all of these conditions too.

(3) The quadratic equation of state $P = k\tilde{\rho}^2$ gives

$$m(v, \mathsf{r}) = c_2 - 2\left(\frac{\mathsf{r}}{2c_1} - \frac{\sqrt{\eta}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}\mathsf{r}}{\sqrt{\eta}}\right)}{2\sqrt{2}c_1^{3/2}}\right), \quad (39)$$

where $\eta = 4k/\kappa$. Making use of Eq. (30), we have

$$2\left[\dot{c}_2 + \frac{\dot{c}_1}{2c_1^2}\right]X_0^2 - X_0 + 2 = 0. \quad (40)$$

If we let $\dot{c}_2 = 0.01$ and $\dot{c}_1 = 0$ (so $\dot{m}_0 > 0$), the above equation reduces to

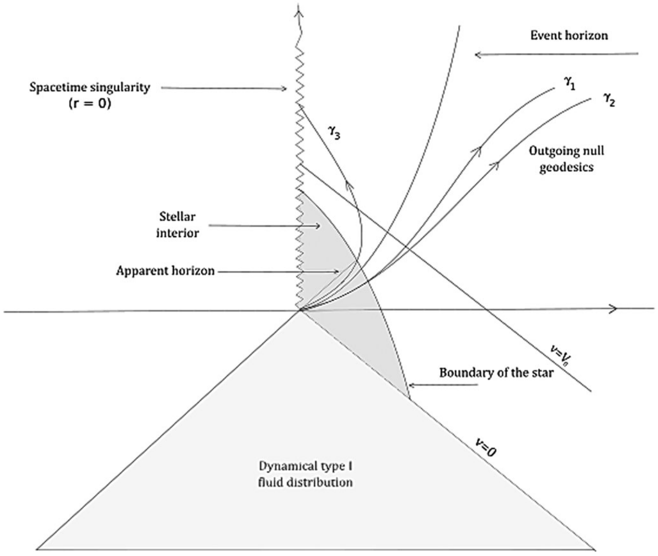

FIG. 3. Quadratic: Here $v = 0$ depicts the first collapsing null ray falling into the central singularity, superposed onto a dynamic and collapsing distribution of type I matter. In this case the naked singularity forms at the origin as before with escaping null geodesic trajectories γ_1 and γ_2. However, due to the form of the mass function (39) for the quadratic equation of state here, we instead have an injected radiation flow into an initially radiated region (consisting of a type I fluid) focused into the central singularity of growing mass, as opposed to a static region in the preceding case (Fig. 2).

$$2(0.01)X_0^2 - X_0 + 2 = 0, \tag{41}$$

which admits two positive real roots. One of these roots is $X_0 = 2.087121525$ so we have a naked singularity. We also have that $c_2(v) = 0.01v$ and $c_1 = \text{const}$, and in Fig. 3 we have a dynamical type I and light-like fluid distribution focused into the central singularity of growing mass forming at the origin. Again, at $v = 0$, a shell of radiation falls through this type I fluid and into the central singularity. In the region where $v > 0$ and for those values of c_1 and c_2 we have the generalized Vaidya solution (39) collapsing to the singularity. Also, $\lim_{\hat{k}\to 0}\hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) = 0.02178 > 0$; thus, this singularity is indeed strong. Finally, this X_0 is also less than $(\frac{dv}{d\mathrm{r}})_{AH} = 50$ so the third condition of *Proposition 1* is satisfied.

(4) The generalized quadratic case $P = k\tilde{\rho}^2 + k_2\tilde{\rho} + k_3$ yields

$$m(v,\mathrm{r}) = -\frac{1}{\eta}\int (\mathrm{r}^2\tan(\sqrt{\zeta}(\ln\mathrm{r} - c_1))\sqrt{\zeta})d\mathrm{r} + c_2, \tag{42}$$

where again $\eta = 4k/\kappa$. We have set $\zeta = k_3\kappa\eta - k_2^2 - 2k_2 - 1$ for convenience. Equation (30) becomes

$$2\dot{c}_2X_0^2 - X_0 + 2 = 0,$$

which, if we set $\dot{c}_2 = 0.01$, becomes identical to Eq. (41). Thus, all of the conditions will be satisfied for this case.

(5) For the polytropic equation of state $P = k\tilde{\rho}^\gamma$, we have a mass function of the form

$$m(v,\mathrm{r}) = \int\left[(\gamma+1)k\kappa\left(\frac{2}{\kappa}\right)^\gamma \times \frac{\mathrm{r}^{2-2\gamma}}{2-2\gamma} + (1-\gamma)c_1\right]^{\frac{1}{1-\gamma}} d\mathrm{r} + c_2. \tag{43}$$

It should be noted that this solution was first presented in Ref. [39]. Using Eq. (30), we have

$$2\dot{c}_2X_0^2 - X_0 + 2[(1-\gamma)c_1]^{\frac{1}{1-\gamma}}X_0 + 2 = 0. \tag{44}$$

If we let $c_1 = 1$, $\dot{c}_2 = 0.01$ with $\gamma > 0$ we have the following

$$2(0.01)X_0^2 - X_0 + 2[(1-\gamma)c_1]^{\frac{1}{1-\gamma}}X_0 + 2 = 0,$$

which is not solvable for any integer $\gamma > 0$. If we let $\gamma = \frac{1}{2}$, Eq. (44) will admit two positive real roots. One of these is $X_0 = 5$, and thus the singularity is naked. Also, we have $\lim_{\hat{k}\to 0}\hat{k}^2\psi = \frac{1}{4}X_0^2(2\dot{m}_0) = \frac{1}{8} > 0$ so the condition for a strong singularity is satisfied. Finally, $(\frac{dv}{d\mathrm{r}})_{AH} = 25 > X_0$ so the final condition is satisfied.

In all the above cases, $c_1 = c_1(v)$ and $c_2 = c_2(v)$ are integration functions. A summary of the algebraic equations for X_0 for each equation of state is presented in Table I.

TABLE I. Equations of tangents X_0 to the singularity curve and values of $\lim_{\hat{k}\to 0}\hat{k}^2\psi$ for several generalized Vaidya mass functions.

Equation of state	Equation for tangent to the singularity curve X_0	$\lim_{\hat{k}\to 0}\hat{k}^2\psi$
Linear	$2X_0^{2k+1} + 2\dot{c}_2X_0^2 - X_0 + 2 = 0$	$\frac{1}{4}X_0^2(2\dot{c}_2)$
Generalized linear	$2X_0^{2k+1} + 2\dot{c}_2X_0^2 - X_0 + 2 = 0$	$\frac{1}{4}X_0^2(2\dot{c}_2)$
Quadratic	$2[\dot{c}_2 + \frac{\dot{c}_1}{2c_1}]X_0^2 - X_0 + 2 = 0$	$\frac{1}{4}X_0^2(2[\dot{c}_2 + \frac{\dot{c}_1}{2c_1}])$
Generalized quadratic	$2\dot{c}_2X_0^2 - X_0 + 2 = 0$	$\frac{1}{4}X_0^2(2\dot{c}_2)$
Polytropic	$2\dot{c}_2X_0^2 - (1 - 2[(1-\gamma)c_1]^{\frac{1}{1-\gamma}}) + 2 = 0$	$\frac{1}{4}X_0^2(2\dot{c}_2)$

VIII. CONCLUSION

In this work we detailed the general mathematical framework to describe the gravitational collapse of a generalized Vaidya spacetime in the context of the CCC. The structure of the central singularity was studied in order to show that it can be a node with outgoing null geodesics emerging from a singular point with a definite value of the tangent, depending on the parameters in the problem and the nature of the generalized Vaidya mass function in question.

We considered a spherically symmetric radiating star. We noted that any astrophysical star is a combination of three distinct concentric zones: the innermost two-component matter zone, the middle radiation zone, and the outermost zone which is the vacuum Schwarzschild exterior. The mass functions obtained in Ref. [15] for various equations of state were analyzed using this mathematical framework and it was shown that in each case, the collapse terminates with a local central singularity which is naked. The strengths of these naked singularities were calculated and it was shown that they are strong curvature singularities and no extension of spacetime through them exists. This has consequences which are far reaching as their presence will no longer make the global spacetime asymptotically simple. This is to say that the theorems of black hole dynamics may require some reformulation. With this, for any realistic mass function, there exists an open set for which the central singularity is naked in the parameter space, and the CCC is violated. That is, the occurrence of a naked singularity is a phenomenon which can be referred to as "stable," despite any changes in the matter field due to a combination of a radiation-like field with a collapsing perfect fluid. With regard to pure type I matter fields, this result is well known [23,40].

It is important to note that during the later stages of gravitational collapse, the generalized Vaidya spacetime is more realistic and physically reasonable than pure dust-like matter or perfect fluid fields. Any collapsing star must radiate and so there exists a combination of a perfect fluid and lightlike matter for this period in the evolution of the star.

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