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Cosmic censorship and charged radiation in second order Lovelock gravity

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ABSTRACT

The conditions for naked singularity formation are considered for a radiating metric of Boulware–Deser type within an electromagnetic field in second order Lovelock (or Einstein–Gauss–Bonnet) gravity. The spacetime metric remains real only up to certain maximum charge contribution. This differs from general relativity. Beyond a certain maximal charge, there exists no real and physical spacetime since the metric becomes complex. We establish that, under certain parameters and for specific values of the mass function and charge contribution, this branch singularity is indeed a naked singularity. This is in contrast to the neutral case where the spacetime metric is always real for a positive mass function, and further, a weak, initially naked singularity always occurs before it becomes covered by an event horizon for all future time. We highlight that both neutral and charged collapse under gravity in Einstein–Gauss–Bonnet gravity differ significantly to their general relativistic counterparts.

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1. Introduction

The study of black holes remains a fruitful and important endeavour in both observational and theoretical astrophysics, general relativity and various modified gravity theories. The theorems of singularity formation predict that spacetime singularities will manifest upon the termination of gravitational collapse [1]. Their incarnation depends on certain circumstances such as causality

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preservation and, specifically, the existence of trapped surfaces. These conditions hold for various gravitational theories, modified or otherwise. One of the conditions for geodesic incompleteness is the existence of trapped surfaces [2]. These theorems only take into account the formation of singularities, and not their nature. The theorems of singularity formation do not consider the possibility of naked singularity formation, i.e. for any escaping null geodesics directed into future infinity. In order to avoid the notion of naked singularities, Penrose [3] envisioned the cosmic censorship conjecture (CCC), divided into the weak CCC and strong CCC, which are yet to be proven.

- *Weak cosmic censorship conjecture (WCCC)*: The weak CCC states that, given any initial generic data, the maximal Cauchy development holds a complete future null infinity. This is to say that if any physically reasonable matter distribution collapses under its own gravity, a spacetime singularity must form which is hidden underneath an event horizon for as long as it exists. Therefore, a black hole is the final fate of gravitational collapse; the event horizon blankets the singularity from all external observers.
- *Strong cosmic censorship conjecture (SCCC)*: The strong CCC states that, given generic asymptotically flat or compact initial data, the maximal Cauchy development is locally inextendible as a regular Lorentzian manifold; singularities are generically spacelike or null, and not timelike.[1] Thus, given initial data, the final fate for all observers should be predictable for all time.

There exists the notion of a *very strong* CCC which asserts that the Lorentzian manifold is continuous as opposed to just being regular; generically, singularities are spacelike. Cauchy surfaces and developments were discussed in Choquet-Bruhat and Geroch [4]; we highlight some definitions in the Appendix. The strongest version of the CCC was disproven by Dafermos and Luk [5] for the Cauchy horizon of a dynamical rotating vacuum black hole. The Vaidya–Papapetrou model [6,7] was an early model proposing a counterexample to cosmic censorship. Other counterexamples for certain matter distributions, in differing contexts also exist, see for example [8–16]. With regards to the preservation of cosmic censorship, Christodolou [17] analysed the Einstein–Klein–Gordon system for spherical gravitational collapse of a scalar field in four dimensional general relativity. It was shown that in that context, cosmic censorship turned out to indeed be a theorem.

Considering general relativity, increasing the spacetime dimension may or may not restore cosmic censorship. It was shown in [12,18,19] that under certain conditions, i.e. certain forms of the mass function, horizon formation takes place in higher dimensions, and so gravitational collapse ceases with the formation of covered singularities. We note that an increase of dimension does not necessarily restore cosmic censorship. For example, Figueras et al. [20] examined the ultraspinning instability of a six dimensional asymptotically flat Myers–Perry black hole. It was demonstrated that this instability implies a naked singularity, and hence, in asymptotically flat higher dimensional spaces, there is a violation of the weak CCC. Further considerations of naked singularities in higher dimensions can be found in [21,22].

Alternative or modified gravity theories are now a common place in the literature and the reasons for modifying conventional Einstein gravity lie in the fact that it is incomplete; it does not, for example, explain the black hole information paradox which deals with the emission of radiation from an isolated black hole. It turns out that it is possible for the Lagrangian action to be of polynomial form [23,24], which leads to the Lovelock action. The quadratic polynomial is the second order Lovelock or Einstein–Gauss–Bonnet (EGB) action. We then have EGB gravity. These quadratic curvature expressions act as corrections to conventional general relativity or first order Lovelock gravity. The higher dimensional EGB-Schwarzschild analogue was first discovered by Boulware and Deser [25] in arbitrary dimensions. The radiating Boulware–Deser solution was studied by Kobayashi [26] and Brassel et al. [27]. It was shown in [28–30] that gravitational contraction yielded naked singularities in five dimensional EGB gravity. The singularity was initially naked and conical in nature before a trapping horizon formed, covering it. We note that for the Boulware–Deser black hole in five dimensions, the mass function at the horizon is given by $M_H(r_H) = r_H^2 + 2\alpha$, which is in terms of the Gauss–Bonnet coupling constant α. Therefore, there exists a mass gap in five

[1] A locally naked singularity is a timelike singularity, so the SCCC asserts that, generically, locally naked singularities are not possible.

dimensions: the mass function does not vanish for a zero radius; it is indeed a function of the Gauss–Bonnet coupling constant, i.e. $M_{r=0} = 2\alpha$. This feature is unique only in five dimensions and does not occur in the higher dimensional case. In higher dimensions, this central singularity is no longer necessarily naked initially. This is a feature which does not occur in Einstein gravity. The metric with an electromagnetic field is very different. The Boulware–Deser metric was studied with an additional charge component by Wiltshire [31,32]. Again, using the methods employed in [26,33–35], the metric can be analysed in Einstein–Gauss–Bonnet–Maxwell (EGBM) gravity as a radiating solution, in which case it reduces to the charged Vaidya metric in the Einstein limit. The final fate of charged radiation collapse in EGB gravity is a singularity which acts like a branch splitting the physically real spacetime from an unphysical complex metric [35]. This singularity is trapped by an inner (Cauchy) horizon and an outer horizon which any external observer can discern. Firstly, this holds for all dimensions $N \geq 5$, unlike the collapse scenario without a charge component, in which an initially naked singularity forms post collapse; this need not happen for dimensions of six and higher. Secondly, this is significantly different to the Einstein limiting case, i.e. charged Vaidya collapse. Thirdly, it is possible that once collapse is completed, the singularity nature will be different to the neutral scenario. Is cosmic censorship adhered to? Are naked singularities possible?

The basis of this paper is to study the environment of the singularities forming upon the cessation of collapse, in the context of the CCC, for the radiating Boulware–Deser spacetime with charge in five dimensions, which has not been done before. We demonstrate that a naked singularity is possible in EGBM gravity, for a particular mass function, charge contribution function, obeying all the energy conditions, and certain parameters. This is fundamentally disparate to the scenario without an electromagnetic field where, upon collapse, a naked singularity will *always* form which is weak and conical in nature [29]. Therefore it is possible to extend spacetime through the singularity; this singularity will be trapped by an event horizon after a time depending on the EGB coupling constant α. Hence we will demonstrate that the effect of the electromagnetic field and the higher order curvature terms profoundly changes the dynamics of collapse and the nature of forming singularities.

2. Einstein-Gauss–Bonnet gravity

In recent times Lovelock gravity, of which EGB gravity and general relativity are special cases, has been studied extensively in many physical contexts, i.e. in the framework of inflation [36–41], Einstein–Scalar–Gauss–Bonnet black holes and wormholes [42,43], and singular bouncing cosmologies [44]. The EGB theory has also proved fruitful in various studies [45,46] of the gravitational wave (GW) signal emanating from the neutron star merging event GW170817. This GW signal was detected by both the LIGO and Virgo detectors in 2017, originating from the shell-elliptical (or lenticular) galaxy NGC 4993. It was shown in [47] that EGB theories can have GW speeds equal to light rendering the analysis compatible with the aforementioned GW170817 event. In the above works, linear and nonlinear functions of the scalar field were non-minimally coupled to the four dimensional Gauss–Bonnet invariant. We remark that EGB gravity is higher dimensional and relevant only for $N \geq 5$. The recent "novel" four dimensional EGB theory as introduced by Glavan and Lin [48] has various fundamental flaws. The "decomposition" of the Lovelock tensor which is required to make four dimensional EGB gravity work leads to a violation of the Bianchi identity, in which case, gravity cannot be coupled to a conserved source [49]. Several authors, including these above, have then considered non-minimally coupling the four dimensional Gauss–Bonnet invariant to a scalar field, which may yield viable models. The curvature corrections indicative of EGB gravity can indeed have implications in observational cosmology and astrophysics, in which case the EGB theory can be considered a viable modified gravity theory.

The action for second order Lovelock (or EGB) gravity is

$$S = \int d^N x \sqrt{-g}(\alpha_1 \mathcal{R} + \alpha_2 \mathcal{R}^2), \tag{1}$$

in arbitrary dimensions, α_1 is the constant affiliated with the action ($\mathcal{R} = R$) of general relativity, and $\alpha_2 = \alpha > 0$ is the EGB coupling constant, which must be positive to avoid pathologies. The second order Lovelock Lagrangian is given by

$$\mathcal{R}^2 = L_{GB} = R^2 + R_{abcd}R^{abcd} - 4R_{cd}R^{cd}, \tag{2}$$

Variation of (1) with respect to g_{ab} will yield the EGB field equations

$$G_{ab} - \frac{\alpha}{2}H_{ab} = \kappa_N T_{ab}. \tag{3}$$

We have that G_{ab} is the Einstein curvature tensor, κ_N is the N-dimensional Einstein constant, T_{ab} is the stress energy tensor and H_{ab} is the Lovelock tensor which is a new term which appears as a consequence of the second order Lagrangian (2) appearing in the action (1). It is given by

$$H_{ab} = g_{ab}L_{GB} - 4RR_{ab} + 8R_{ac}R^{c}{}_{b} + 8R_{acbd}R^{cd} - 4R_{acde}R_{b}{}^{cde}. \tag{4}$$

In EGB gravity, $N = 5$ and $N > 5$ are considered as the critical dimensions. If the spacetime dimension is $N < 5$, the Lovelock tensor $H_{ab} = 0$ identically, and we have general relativity in four dimensions or Newtonian gravity for dimensions $N < 4$. In the limit of vanishing α, five dimensional Einstein gravity will result.

3. Metric and field equations

We note that in arbitrary spacetime dimensions N, we have that Einstein's coupling constant is given by

$$\kappa_N = \frac{2(N-2)\pi^{\frac{N-1}{2}}}{(N-3)\Gamma\left(\frac{N-1}{2}\right)}, \tag{5}$$

which contains the gamma function. The total area covering the outer surface of the $(N-2)$-sphere is calculated from

$$\mathcal{A}_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}, \tag{6}$$

where we note the explicit dependence on the dimension N. The electromagnetic energy tensor takes the form

$$E_{ab} = \frac{1}{\mathcal{A}_{N-2}}\left(F_{a}{}^{c}F_{bc} - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right), \tag{7}$$

and has a zero trace only in four dimensions. In the above, the Faraday tensor $F_{ab} = \Phi_{b;a} - \Phi_{a;b}$ where Φ is the N-potential. In five dimensions, the coupling constant and surface area are

$$\kappa_5 = 3\pi^2, \qquad \mathcal{A}_3 = 2\pi^2, \tag{8}$$

respectively.

The five dimensional Boulware–Deser metric is given by

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\Omega_3^2, \tag{9}$$

where $d\Omega_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2$ and

$$f(v,r) = 1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M(v,r)}{r^4}}\right). \tag{10}$$

In the above, $M(v,r)$ is the general five dimensional gravitational mass. For a generalized two-component type II distribution containing a dichotomy of null dust and a null string fluid, the energy momentum tensor is

$$T_{ab} = \mu l_a l_b + (\rho + P)(l_a n_b + l_b n_a) + P g_{ab}, \tag{11}$$

where we have that

$$l_a = \delta^0{}_a, \qquad n_a = \frac{1}{2}f(v,r)\delta^0{}_a + \delta^1{}_a, \tag{12}$$

with the following restrictions for the null vectors $l_c l^c = n_c n^c = 0$, $l_c n^c = -1$. Here we have that μ is the energy density of the null dust and ρ and P are the energy density and pressure of the null string.

After a lengthy calculation, the EGB field equations (3) take on the simple forms

$$\mu = \frac{M_v}{2\pi^2 r^3}, \tag{13a}$$

$$\rho = \frac{M_r}{2\pi^2 r^3}, \tag{13b}$$

$$P = -\frac{M_{rr}}{6\pi^2 r^2}, \tag{13c}$$

where subscripts indicate differentiation. If we make the selection

$$\begin{aligned} M(v,r) &= M(v) - \frac{\kappa_5 \mathcal{Q}(v)^2}{6\mathcal{A}_3 r^2} \\ &= M(v) - \frac{\mathcal{Q}(v)^2}{4r^2}, \end{aligned} \tag{14}$$

for the mass function in (10) we have that the charge contribution (7) is then embedded into the definition (14). Note that $\mathcal{Q} = \mathcal{Q}(v)$ is the benefaction of charge for EGBM theory. We then arrive at the charged Boulware–Deser metric [31]

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\Omega_3^2, \tag{15}$$

where

$$f(v,r) = 1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M(v)}{r^4} - \frac{2\alpha\mathcal{Q}(v)^2}{r^6}}\right). \tag{16}$$

In the general relativity limit, this metric reduces to the charged Vaidya one in five dimensions.

With the aid of (13), the EGBM field equations can be written as

$$\mu = \frac{M_v}{2\pi^2 r^3} - \frac{\mathcal{Q}\mathcal{Q}_v}{4\pi^2 r^5}, \tag{17a}$$

$$\rho = P = \frac{\mathcal{Q}^2}{4\pi^2 r^6} \tag{17b}$$

When $\mathcal{Q}(v) = 0$, then $M(v,r) = M(v)$ from (14), and only one field equation (17a) will remain. For the spacetime (15) with metric (16), we must have that the energy conditions hold:

$$M_v \geq \frac{\mathcal{Q}\mathcal{Q}_v}{2r^2}, \qquad\qquad \mathcal{Q}(v) \geq 0.$$

Note that since the term $\frac{2\alpha\mathcal{Q}^2}{r^6}$ has a negative sign in (16), there is a maximum charge endowment that exists before the metric ceases to be real [35].

4. Gravitational collapse

4.1. Singularity type

There are some marked differences between the gravitational collapse process of the charged Boulware–Deser spacetime (16) and the uncharged counterpart (10). The collapse dynamics of the conventional Boulware–Deser spacetime are precariously affected by the constant α. Firstly, the spacetime metric is well defined and regular for all r, however the manifold on which the metric sits is itself singular because of the diverging Kretschmann invariant [29]. Therefore collapse ceases with a central *weak and extended conical singularity*. Secondly, the EGB constant α impedes the horizon formation for a time period, therefore the singularity remains naked initially. The trapping horizon eventually forms creating a black hole containing the trapped surfaces and conical singularity. Charged Boulware–Deser collapse is fundamentally contrasting. Firstly, the metric is no longer well defined and is singular near $r = 0$; in fact there is no spacetime at $r = 0$. The charge benefaction *refashions* the type of singularity encountered once collapse is completed; it ends with the creation of a *strong curvature branch-like singularity* separating the two regions from each other [35]. This branch singularity is *a fortiori* the result of the maximal contribution of charge keeping the square root in (16) real. Consider the square root

$$\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha \mathcal{Q}(v)^2}{r^6}}=0.$$

This can be written as

$$r_s^6+8\alpha M r_s^2-2\alpha\mathcal{Q}^2\geq 0. \tag{18}$$

The above inequality has six solutions, four complex and two real, one of which is positive. If $\mathcal{Q} \neq 0$ and $M > 0$, for this positive root, there is a branch-like singularity $r = r_s(v)$ which separates the complex metric from the physical spacetime. The value of the above inequality gives $0 < r_s < r_1 \leq r_0 < \infty$ as the domain of r; there is *no* spacetime in the interval $0 < r_s$. The Cauchy and outer horizons (r_1 and r_0 respectively) form at $v = 0$, $r_s > 0$ which was not the case for $\mathcal{Q} = 0$. The constant α along with the charge contribution $\mathcal{Q}$ significantly change the collapse dynamics. We note that only the outer horizon is observable from infinity, and contains the inner horizon and trapped surfaces. Fig. 1 showcases the behaviour of the Boulware–Deser metric (16) for both the charged and neutral cases (we have used $\alpha = 2$, $v = 2$ and positive values for $M(v)$ and $\mathcal{Q}(v)$). It can clearly be seen that for $\mathcal{Q}(v) \neq 0, f(v, r)$ is not well defined at $r = 0$ since this is the region where the metric becomes complex by expression (18). When $\mathcal{Q}(v) = 0$, the metric function is everywhere well defined for all $\alpha > 0$. The shaded region simply depicts the values of r for which both cases are mutually well defined. We note that for $\mathcal{Q}(v) \neq 0$, the metric functions vanish for two values of r which would correspond to the formation of two horizons. When $\mathcal{Q}(v) = 0$, the metric $f(v, r)$ vanishes for one value of r which would indicate a single event horizon. It can also be seen that for larger values of r, the two cases appear to coincide which implies that the charged and neutral scenarios are indistinguishable from each other, for any observer at very large distances from the source.

In the lower order limit of string theory it is known that the Gauss–Bonnet constant α is very small, therefore it can be interpreted, in regimes where $K\alpha \ll 1$ ($K = R_{abcd}R^{abcd}$ is the Kretschmann invariant) that the second order Lovelock term $\mathcal{R}^2 = L_{GB}$ presents as a correction to the theory. This notion is similar for the action of general relativity. If we suppose that $\alpha \ll 1$, i.e. very small, we can perform a second order Taylor expansion, for example, on the metric function (16) which yields

$$f(v,r)\approx 1-\frac{M}{r^2}+\frac{Q^2}{4r^4}+\left(\frac{(4Mr^2-Q^2)^2}{8r^{10}}\right)\alpha+\mathcal{O}(\alpha^3), \tag{19}$$

which is then the perturbed EGB metric. The first three terms $1-\frac{M}{r^2}+\frac{Q^2}{4r^4}$ are indeed those of the conventional five dimensional Reissner–Nordström metric. Therefore for very small α, the term $\left(\frac{(4Mr^2-Q^2)^2}{8r^{10}}\right)\alpha$ is the perturbative EGB correction to the Reissner–Nordström metric. Thus, the Gauss–Bonnet term $\mathcal{R}^2 = L_{GB}$ in the action (1) acts, in a sense, as a perturbative quadratic correction to the Einstein–Hilbert action.

Since uncharged Boulware–Deser collapse results in an (at least) initially naked singularity, the next questions arise: is it possible that there is *no formation* of the above mentioned horizons? Are naked singularities possible in five dimensional charged Boulware–Deser collapse?

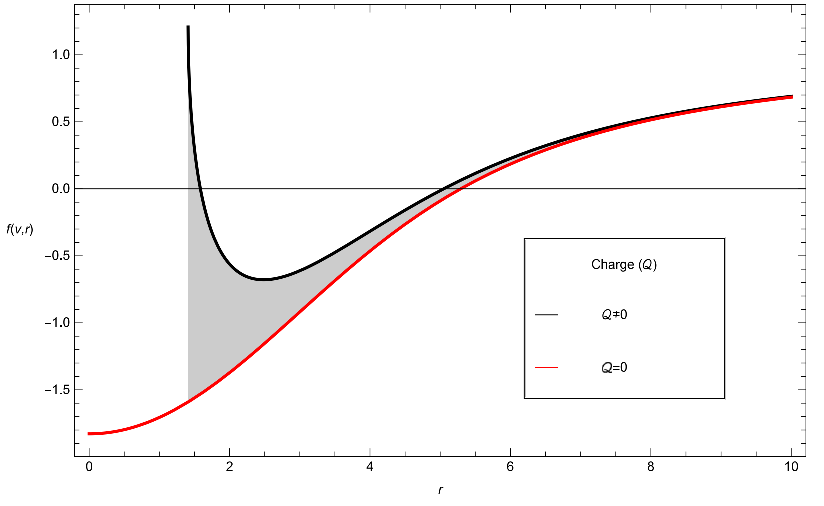

Fig. 1. A visual representation of the evolution of the metric function (16) against r for $\mathcal{Q}(v) \neq 0$ and $\mathcal{Q}(v) = 0$. The charged metric function $f(v,r)$ is not defined at $r = 0$ and the shaded region indicates the values of r for which both cases are well defined, mutually. For larger values of r both cases appear to coincide.

4.2. Model of collapse: singularity analysis

We now study the gravitational collapse of charged radiation described by the metric (16). For an EGB universe which is asymptotically flat and empty at infinite distances, a thick shell of radiation surrounded by an electromagnetic field contracts at the centre of symmetry [50]. If K^a is the tangent to the nonspacelike geodesics where $K^a = \frac{dx^a}{dk}$, then $K^a{}_{;b}K^b = 0$ and

$$g_{ab}K^aK^b = \mathcal{B}, \tag{20}$$

where k is an affine parameter and $\mathcal{B}$ is some constant which describes geodesic classes. When $\mathcal{B} = 0$, this characterizes null geodesics while $\mathcal{B} < 0$ implies timelike geodesics. The expressions $\frac{dK^v}{dk}$ and $\frac{dK^r}{dk}$ can be calculated (using the Euler–Lagrange equations)

$$\frac{\partial L}{\partial x^a} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0, \tag{21}$$

where we have the usual Lagrangian $L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$. For the charged spacetime (16) these equations become, after a lengthy calculation,
v-component:

$$\begin{aligned}\frac{dK^v}{dk} = -\frac{1}{2}\Bigg[&\frac{r}{2\alpha}\left(1-\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha\mathcal{Q}(v)^2}{r^6}}\right) \\ &-\frac{r^2}{8\alpha}\left(\frac{\frac{12\alpha\mathcal{Q}}{r^7}-\frac{32\alpha M}{r^5}}{\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha\mathcal{Q}(v)^2}{r^6}}}\right)\Bigg](K^v)^2 \\ &+r\left((K^\theta)^2+\sin^2\theta(K^\phi)^2+\sin^2\theta\sin^2\phi(K^\psi)^2\right).\end{aligned} \tag{22}$$

r-component:

$$\frac{dK^r}{dk} = -\Bigg[\frac{r}{2\alpha}\left(1-\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha\mathcal{Q}(v)^2}{r^6}}\right)$$

$$
\begin{aligned}
&-\frac{r^2}{8\alpha}\left(\frac{\frac{12\alpha\mathcal{Q}}{r^7}-\frac{32\alpha M}{r^5}}{\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha\mathcal{Q}(v)^2}{r^6}}}\right)\Bigg]\left(\frac{1}{2}f(v,r)(K^v)^2-K^vK^r\right)\\
&+\frac{r^2}{16\alpha}\left(\frac{\frac{8\alpha M_v}{r^4}-4\alpha\frac{\mathcal{Q}\mathcal{Q}_v}{r^6}}{\sqrt{1+\frac{8\alpha M(v)}{r^4}-\frac{2\alpha\mathcal{Q}(v)^2}{r^6}}}\right)(K^v)^2\\
&+f(v,r)r\left((K^\theta)^2+\sin^2\theta(k^\phi)^2+\sin^2\theta\sin^2\phi(K^\psi)^2\right). \qquad (23)
\end{aligned}
$$

θ-component:

$$
\frac{dK^\theta}{dk}+\frac{2}{r}K^rK^\theta-\sin\theta\sin\phi\left((K^\theta)^2+\sin^2\phi(K^\psi)^2\right)=0. \qquad (24)
$$

ϕ-component:

$$
\frac{dK^\phi}{dk}+\frac{2}{r}K^rK^\phi+2\cot\theta K^\theta K^\phi-\sin\phi\cos\phi(K^\psi)^2=0. \qquad (25)
$$

ψ-component:

$$
\frac{dK^\psi}{dk}+\frac{2}{r}K^rK^\psi+2\cot\theta K^\theta K^\psi+2\cot\phi K^\phi K^\psi=0. \qquad (26)
$$

In the expression (23) above, $f(v,r)$ is the function from (16). Following the approaches of [51] we can write

$$
K^v=\frac{P}{r}, \qquad (27)
$$

where we have that $P=P(v,r)$ is an arbitrary function. Using Eqs. (22) and (23) and noting that $\mathcal{B}=g_{ab}K^aK^b$, a lengthy calculation yields

$$
K^v=\frac{dv}{dk}=\frac{P}{r}, \qquad (28a)
$$

$$
K^r=\frac{dr}{dk}=f(v,r)\frac{P}{2r}+\frac{\mathcal{B}r}{2P}-\frac{l^2}{2rP}. \qquad (28b)
$$

In the above, l is the impact parameter.

5. Locally naked singularity conditions

We will now examine whether the end state of charged radiation collapse in EGB gravity is a naked singularity or a black hole. We note that there is no spacetime at $0<r_s$, yet the branch singularity (18) is the curvature singularity of the spacetime [35]. For a charged shell of radiation with a large enough mass, this branch singularity begins to form at $r_s=v=0$ and extends into the future. If there exist families of trajectories directed into the future reaching observers at infinity in the spacetime, the singularity which forms post collapse will then be considered naked. A charged black hole will result if no such trajectories exist, and the two horizons form sufficiently early.

5.1. Existence of outgoing nonspacelike geodesics

If we *a priori* allow X_0 to be a limiting value at $r_s=v=0$, i.e. the tangent to the radial geodesic, on any singular geodesic, the nature of this limiting value is calculated as

$$
X_0=\lim_{r_s=v\to 0}X=\lim_{r_s=v\to 0}\frac{v}{r_s}. \qquad (29)
$$

Using the above expression (29) an explicit equation for X_0 can be found which will dictate the behaviour of all null geodesics in the region of the singularity r_s. Differentiating (18) yields

$$
\frac{dr_s}{dv}+\frac{4}{3}\alpha\frac{1}{r_s^3}\frac{dM}{dv}+\frac{8}{3}\alpha M\frac{1}{r_s^4}\frac{dr_s}{dv}-\frac{2}{3}\alpha\mathcal{Q}\frac{1}{r_s^5}\frac{d\mathcal{Q}}{dv}=0. \qquad (30)
$$

For a well defined tangent at the singularity r_s to exist in the above expression, the mass function $M(v) \sim \lambda v^4$ and the charge function $\mathcal{Q}(v) \sim \beta v^3$, where λ and β are positive real constants. We can then write Eq. (30) as

$$\frac{dr_s}{dv} + \frac{16}{3}\alpha\lambda\left(\frac{v}{r_s}\right)^3 + \frac{8}{3}\alpha\lambda\left(\frac{v}{r_s}\right)^4\frac{dr_s}{dv} - 2\alpha\beta^2\left(\frac{v}{r_s}\right)^5 = 0. \tag{31}$$

The choices $M(v) \sim \lambda v^4$ and $\mathcal{Q}(v) \sim \beta v^3$ in (30) are the only choices that will allow for the full invocation of (29) into the resulting Eq. (31). The above then reduces to the following

$$X_0^6 - \frac{4\lambda}{\beta^2}X_0^4 - \frac{1}{2\alpha\beta^2} = 0, \tag{32}$$

using (29). The above algebraic equation needs to be solved to determine the nature of the singularity. The equation for null geodesics for the spacetime metric (15) is

$$X_N = \frac{K^v}{K^r} = \frac{dv}{dr} = \frac{2}{1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M(v)}{r^4} - \frac{2\alpha\mathcal{Q}(v)^2}{r^6}}\right)}, \tag{33}$$

which reduces to

$$X_N = 2, \tag{34}$$

in the vicinity of the singularity $r_s = v = 0$, since the term in brackets vanishes.

5.2. Sufficient conditions

We begin by stating the following Lemma:

Lemma 5.1. *If the functions for the mass and charge obey $M(v) \sim \lambda v^4$ and $\mathcal{Q}(v) \sim \beta v^3$, where λ, β are positive constants, the central singularity at $r_s = v = 0$ is not trapped.*

Proof. Consider the metric (16) with mass and charge functions $M(v) \sim \lambda v^4$ and $\mathcal{Q}(v) \sim \beta v^3$, respectively. We then have

$$f(v,r) = 1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha\lambda v^4}{r^4} - \frac{2\alpha\beta^2 v^6}{r^6}}\right).$$

The metric (16) becomes complex for any positive mass and charge contribution, in general by expression (18). Therefore for these particular choices of $M(v)$ and $\mathcal{Q}(v)$, the same will hold since these choices are never negative. Therefore, for these functional values of $M(v)$ and $\mathcal{Q}(v)$, the central singularity at $r_s = v = 0$ cannot be trapped. □

We can finally state the sufficient conditions [11] determining the extant central singularity which is locally naked for a contracting charged Boulware–Deser spacetime.

Proposition 5.1. *Consider a C^2 Boulware–Deser spacetime undergoing collapse with a mass function $M(v) = \lambda v^4$ satisfying all energy conditions, surrounded by an electromagnetic field with charge contribution $\mathcal{Q}(v) = \beta v^3$ from a regular epoch. If the following conditions are satisfied:*

1. *The partial derivatives of the positive mass $M(v)$ and charge $\mathcal{Q}(v)$ functions are continuous and exist at the central singularity,*
2. *There exist one or more positive and real roots X_0 to the equation*

$$X_0^6 - \frac{4\lambda}{\beta^2}X_0^4 - \frac{1}{2\alpha\beta^2} = 0,$$

3. *At least one real and positive root is less than*

$$X_N = 2,$$

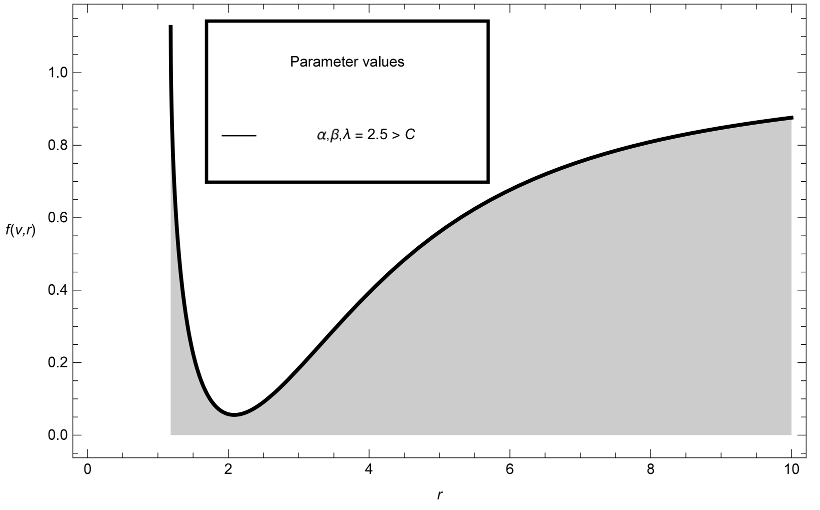

Fig. 2. A plot showing the evolution of the metric function (16) against r for $\mathcal{Q}(v) \neq 0$ and the parameter values $\alpha = \beta = \lambda = 2.5 > C$. It is clear that when the metric $f(v,r) = 0$, it does not coincide to any value of r on the axis, further demonstrating that there is a naked singularity. The shaded region indicates the values of r for which the metric is defined.

at the branch singularity, then the singularity is naked locally and there exists outgoing radial C^1 null geodesics escaping to future infinity.

6. Cosmic censorship

We now consider Eq. (32). It admits six roots, four complex and two real, one of which is positive. This positive and real root is given by

$$X_0 = \frac{1}{\sqrt{6}}\sqrt{\sqrt[3]{2}G + \frac{32\lambda^2 2^{2/3}}{\beta^4 G} + \frac{8\lambda}{\beta^2}}, \tag{35}$$

where

$$G = \sqrt[3]{\frac{3\sqrt{3}\beta^2\sqrt{512\alpha\lambda^3 + 27\beta^4} + 256\alpha\lambda^3 + 27\beta^4}{\alpha\beta^6}}. \tag{36}$$

We need only show that this positive real root is less than X_N. Since α, β and λ are positive constants, we can choose sufficient values to evaluate X_0. If we make the selection $\alpha = \beta = \lambda = C$, say, where C is a real constant, it turns out that the minimum value this constant can take on such that $X_0 < X_N$ is

$$\begin{aligned} C &= \frac{1}{24}\sqrt[3]{566 + 42\sqrt{33}} + \frac{8}{3\sqrt[3]{566 + 42\sqrt{33}}} + \frac{1}{3} \\ &\approx 1.00077. \end{aligned} \tag{37}$$

It turns out that Proposition 5.1 is satisfied if and only if $\alpha = \beta = \lambda > C$, where C is given by (37) since the value of $X_0 < X_N$. If $\alpha = \beta = \lambda < C$ the two event horizons will form blanketing the singularity within the confines of a charged black hole. Fig. 2 depicts the charged metric (16) versus r for the mass and charge functions $M(v) = \lambda v^4$ and $\mathcal{Q}(v) = \beta v^3$ with parameter values $\alpha = \beta = \lambda = 2.5 > C$. We clearly see that the metric $f(v,r)$ no longer intersects with the r-axis (unlike in Fig. 1) indicating that no horizon forms for these above mentioned parameter values, and

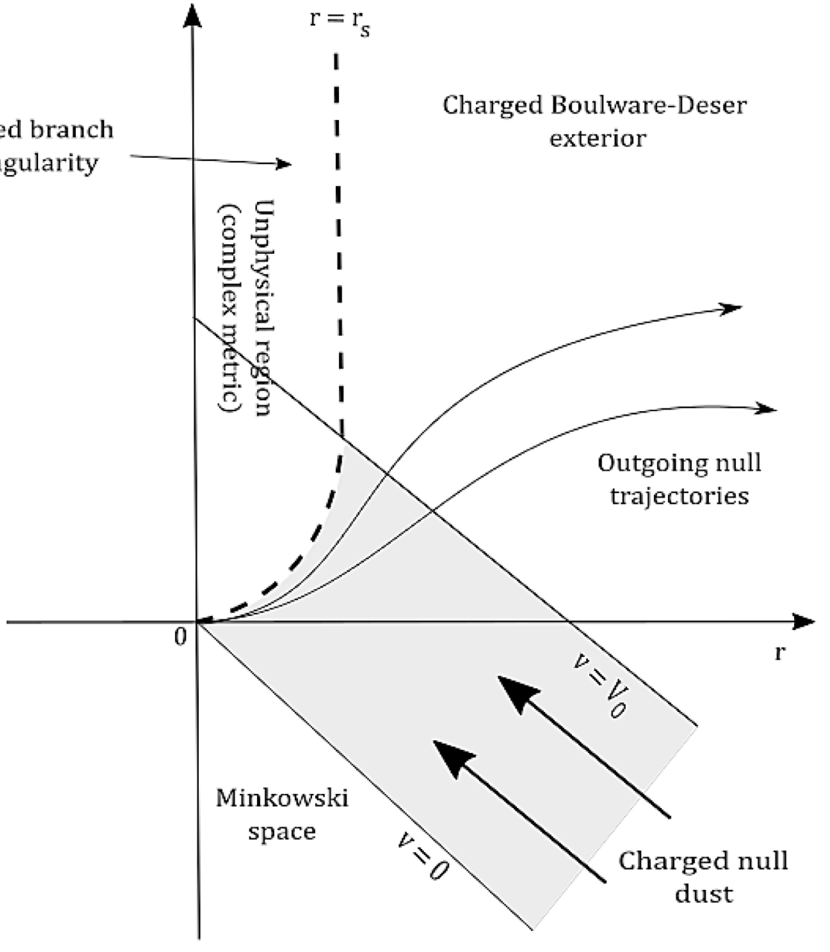

Fig. 3. Spacetime diagram depicting the null radiation collapse process in five dimensional EGBM gravity. There exists a branch singularity $r = r_s(v)$ which forms at $v = r_s(0) = 0$ and extends into the future separating the complex region from the rest of the real contracting spacetime. A naked singularity forms at the origin and there are null geodesic trajectories escaping to infinity. There is an injected and charged flow of null radiating matter into a region initially consisting of a type II fluid with $M = M(v)$ focused into the singularity of growing mass at the centre.

the branch singularity separating the complex metric from the physical spacetime is, in principle, visible to an observer in the external universe; it is a naked singularity.

Various values of α, β and λ as well as the resulting values of X_0 are presented in Table 1.

Therefore, we can state all of the above in the form of a theorem:

Theorem 6.1. *Consider a collapsing five dimensional radiating Boulware–Deser spacetime within an electromagnetic field from a regular epoch, with a positive and real mass function $M(v) = \lambda v^4$ and charge contribution $\mathcal{Q}(v) = \beta v^3$, satisfying all the energy conditions, and which are at least C^2 in the entire spacetime. Should the parameter values of $\alpha = \beta = \lambda > C$, where C is given by (37), the final outcome of gravitational collapse is a central naked singularity.*

Fig. 3 shows the collapse scenario which is possible, from an initial space of Minkowski type, for null charged matter. Radiating charged null matter cascades into a naked singularity. The charge

Table 1
Parameters for naked singularity formation.

Values of α, β, λ	X_0 value	Naked singularity $X_0 < X_N (= 2)$
$\alpha = \beta = \lambda = 1$	$X_0 = 2.0077$	No
$\alpha = \beta = \lambda = C$	$X_0 = 2$	No
$\alpha = \beta = \lambda = 1.5$	$X_0 = 1.6393$	Yes
$\alpha = \beta = \lambda = 2$	$X_0 = 1.41196$	Yes
$\alpha = \beta = \lambda = 3$	$X_0 = 1.1591$	Yes

contribution $\frac{2\alpha Q(v)^2}{r^6}$ in the metric function (16) stipulates that the metric becomes complex for a specific value of $r = r_s$ and there exists *no physical spacetime* below this value. This promotes the creation of a branch naked singularity, for our parameter values, which separates the complex region of the spacetime metric from the real contracting spacetime. The naked singularity forms at $v = r_s(0) = 0$ and extends into the future as per Theorem 6.1. The collapse process will eventually cease at a later time $v = V_0$, with the singularity being visible to external observers in the charged Boulware–Deser exterior.

7. Singularity strength

Supposing we have a null affine parameter $\hat{k}$, we can compute the strength of the singularity if we consider the null geodesics which are parametrized by $\hat{k}$, terminating at the shell-focusing branch singularity $r_s = v = \hat{k} = 0$ [52]. A measure of the destructive capacity of the singularity lies in its strength; is it possible to extend spacetime through it or not? In the case of five dimensional neutral collapse, the singularity is weak and conical and so, in principle, an extension of spacetime through it is indeed possible. Following [11,53], a singularity would be considered strong if the following

$$\lim_{\hat{k}\to 0} \hat{k}^2\eta = \lim_{\hat{k}\to 0} \hat{k}^2 R_{ab}K^aK^b > 0, \tag{38}$$

holds true, where R_{ab} is the Ricci curvature tensor. For our spacetime (15) with function (16), and the choices $M(v) = \lambda v^4$ and $\mathcal{Q}(v) = \beta v^3$, it can be shown after some calculation that the scalar $\eta = R_{ab}K^aK^b$ is given by

$$\eta = \frac{3}{4}\frac{8\lambda X_0^3 - 3\beta^2 X_0^5}{\sqrt{1 + 8\alpha\lambda X_0^4 - 2\alpha\beta^2 X_0^6}}\left(\frac{P}{r}\right)^2, \tag{39}$$

therefore

$$\hat{k}^2\eta = \frac{3}{4}\frac{8\lambda X_0^3 - 3\beta^2 X_0^5}{\sqrt{1 + 8\alpha\lambda X_0^4 - 2\alpha\beta^2 X_0^6}}\left(\frac{P\hat{k}}{r}\right)^2. \tag{40}$$

Evaluating the limit at $\hat{k} \to 0$ yields

$$\lim_{\hat{k}\to 0} \hat{k}^2\eta = \frac{3}{4}\frac{8\lambda X_0^4 - 3\beta^2 X_0^6}{\sqrt{1 + 8\alpha\lambda X_0^4 - 2\alpha\beta^2 X_0^6}}. \tag{41}$$

This condition above depends on the positive and real root X_0, along with the positive parameters α, β and λ. If

$$\lim_{\hat{k}\to 0} \hat{k}^2\eta > 0, \tag{42}$$

we can then establish that the observed naked singularity is strong. In order for the above condition (42) to be satisfied, however, the positive real root X_0 needs to satisfy

$$X_0 = \frac{2\sqrt{6}\sqrt{\lambda}}{3\beta}.$$

Substituting (35) into the above expression will yield the relationship between the constants α, β and λ that can give rise to a strong curvature singularity.

8. Discussion

In this article we analysed the nature of singularities forming from the gravitational collapse of the charged Boulware–Deser spacetime found by Wiltshire [31], in the context of the CCC in five dimensions. The type of singularity forming after collapse terminates is a branch-like singularity $r = r_s(v)$ resulting from the fact that whenever the inequality

$$r_s^6 + 8\alpha M r_s^2 - 2\alpha \mathcal{Q}^2 \geq 0,$$

is violated, the metric function is complex. We showed that, for a particular mass function $M(v)$ and charge subsidy $\mathcal{Q}(v)$, obeying the energy conditions, a naked singularity is indeed possible in EGBM gravity for certain parameters. The central singularity structure was studied in order to show that it can become a node with null and escaping geodesics emanating from a singular point with an assured tangent value, depending on the above mentioned parameters. This is dissimilar to the five dimensional neutral case analysed by [28–30] where a weak and conical naked singularity will *always* form *for any mass function*, post collapse. This singularity eventually succumbs to an event horizon after a time period depending on the Gauss–Bonnet coupling constant α. This feature of uncharged collapse is only prevalent in five dimensions, however; for all $N \geq 6$ collapse need not cease with an initially naked singularity. The *charge contribution* in tandem with the curvature corrections, therefore play a paramount role in the collapse dynamics in EGB gravity. The metric with the charge $\mathcal{Q}(v)$ is always singular and there appears to be no real difference between the $N = 5$ and $N \geq 6$ cases. We derived the conditions for which this naked singularity will be strong; in this case spacetime cannot be extended through it. This is unlike the five dimensional uncharged collapse scenario where the singularity forming is always weak; in principle, spacetime can always be extended through it [29]. Analogously to the general relativity cases studied by [12], higher dimensions may perhaps restore cosmic censorship under certain conditions in the presence of an electromagnetic field in EGB gravity theory.

CRediT authorship contribution statement

Byron P. Brassel: Conceptualization, Methodology, Writing, Figures, Software. **Rituparno Goswami:** Conceptualization, Writing, Editing. **Sunil D. Maharaj:** Conceptualization, Methodology, Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Mathematical preamble for cosmic censorship

The physical behaviour of gravitational curvature singularities remains unknown. The Hawking–Penrose singularity theorems predict that singularities are inevitable upon the collapse of any physically reasonable matter distribution [1]. Excluding finite patches of spacetime hidden underneath event horizons covering singularities, general relativity describes the universe as being deterministic; knowing its state at any moment in time, it is possible to predict the entire evolution of the universe. The failure of cosmic censorship implies the failure of determinism; in the causal future of a spacetime curvature singularity, it is impossible to predict the behaviour of spacetime. We now provide some classical definitions and a theorem which are required for the understanding of cosmic censorship [2,3,5,54].

Definition A.1 (*Cauchy Surface*)**.** Let $(\mathcal{M}, g)$ be a smooth Lorentzian (spacetime) manifold. A Cauchy surface Σ is an embedded submanifold $\Sigma \hookrightarrow \mathcal{M}$ such that every inextendible, differentiable timelike curve in $\mathcal{M}$ intersects Σ precisely at one point.

We note that a Cauchy surface is a spacelike hypersurface on the manifold $\mathcal{M}$ with dimension one less than that of $\mathcal{M}$. Any causal curve without an endpoint passing through any event in the spacetime $\mathcal{M}$ will necessarily intersect the Cauchy hypersurface Σ. The above-mentioned evolution of the universe can be determined by knowing its condition everywhere on the hypersurface Σ at any moment of time.

Definition A.2 (*Global Hyperbolicity*)**.** A Lorentzian manifold $(\mathcal{M}, g)$ which admits a Cauchy surface Σ is called globally hyperbolic.

Global hyperbolicity implies that solutions of the Einstein field equations are uniquely determined from the initial data set, i.e. in this case the Cauchy surface Σ. We are now in the position to state the following theorem [4,54]:

Theorem A.1 (*Maximal Cauchy Development*)**.** *Let a Cauchy surface $(\Sigma, \bar{g}, \mathcal{K})$ be a smooth vacuum initial data set, where $\bar{g}$ and $\mathcal{K}$ are the induced first and second fundamental forms respectively. There exists a unique smooth spacetime $(\mathcal{M}, g)$ such that*

1. *The Ricci scalar vanishes on $\mathcal{M}$, i.e. $R = 0$,*
2. *$(\mathcal{M}, g)$ is globally hyperbolic,*
3. *Any other smooth spacetime with the first two properties isometrically embeds into $\mathcal{M}$.*

The spacetime $(\mathcal{M}, g)$ is then called a maximal Cauchy development.

It should be noted that similar theorems can be proven for other general relativistic systems, for example the Einstein–Maxwell equations, null dust and Einstein-scalar fields.

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