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AN INERTIAL ITERATIVE METHOD FOR SOLVING SPLIT MONOTONE INCLUSION PROBLEMS IN HILBERT SPACES

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Abstract. The purpose of this work is to introduce and study a new type of a relaxed extrapolation iterative method for approximating the solution of a split monotone inclusion problem in the framework of Hilbert spaces. More so, we establish a strong convergence theorem of the proposed iterative method under the assumption that the set-valued operator is maximal monotone and the single-valued operator is Lipschitz continuous monotone which is weaker assumption unlike other methods in which the single-valued is inverse strongly monotone. We emphasize that the value of the Lipschitz constant is not required for the iterative technique to be implemented, and during computation, the Lipschitz continuity was not used. Lastly, we present an application and also some numerical experiments to show the efficiency and the applicability of our proposed iterative method.

1. **Introduction.** Let H be a real Hilbert space with an induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The Monotone Inclusion Problem (MIP) is defined as:

$$\text{Find } x \in H \text{ that solves } \ 0 \in (A + A_1)(x), \tag{1}$$

where $A : H \to H$ and $A_1 : H \to 2^H$ are monotone operators. It is well known that if $A_1 = N_C$ is the normal cone of some nonempty closed and convex subset C of H, then problem (1) becomes the classical Variational Inequality Problem (VIP) (see [15, 16, 28]). There are several problems in the real world that can be formulated as the problem (1). It is important in many different types of mathematical optimizations problems, including variational inequalities problems, minimization problems, linear inverse problems, saddle-point problems, fixed-point problems, split feasibility problems, Nash equilibrium problems in non-cooperative games, and many others (see [6, 11–14, 29] and the references therein). Due to the fruitful applications of problem (1), several authors have introduced and studied different

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iterative techniques to solve problem (1). Among many others, the simplest iterative technique for solving problem (1) is the well known forward-backward splitting technique (see [13, 23]). The iterative technique is defined as

$$\begin{cases} x_0 \in H \\ x_{n+1} = J_{\lambda}^{A_1}(x_n - \lambda A x_n), \end{cases} \tag{2}$$

where $\lambda > 0$ and $J_{\lambda}^{A_1} := (I + \lambda A)^{-1}$. The iterative technique converges weakly to a solution provided that A is α-inverse strongly monotone. In the iterative technique (2), the individual steps within each iteration involve forward evaluations in which the value of the single-valued operator is computed and the backward evaluations in which the re-solvent of the set-valued operator is computed rather than their sum directly. In addition, the Tseng in [31], introduced and studied a modified forward-backward splitting technique. The method is defined as follows.

$$\begin{cases} x_0 \in H \\ y_n = J_{\lambda_n}^{A_1}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n (A y_n - A x_n), \end{cases} \tag{3}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$. It is well known that the Lipschitz constant of an operator is often unknown or very difficult to find or estimate depending on how the operator is being defined. This condition makes the iterative technique difficult to apply to real life problems.

An interesting generalization of the MIP was introduced and studied by Moudafi in [24]. Moudafi [24], introduced and studied the Split Monotone Inclusion Problem (SMIP). The problem is defined as

$$\text{Find } \ x \in H_1 \text{ that solves } 0 \in (A + A_1)(x) \tag{4}$$

such that

$$y = Tx \in H_2 \text{ that solves } 0 \in (B + B_1)(y), \tag{5}$$

where $A : H_1 \to H_1, B : H_2 \to H_2$ are single valued operators, $A_1 : H_1 \to 2^{H_1}, B_1 : H_2 \to 2^{H_2}$ are multi-valued operators and $T : H_1 \to H_2$ is a bounded linear operator. Furthermore, a number of real-world problems can be mathematically represented as SMIP (4) and (5), including signal processing, image restoration, sensor networks, computer tomography, data compression, linear inverse problems, and machine learning (see, for example, [11, 13, 15, 24] and the references therein). It is well known that if $A_1 = N_C$ and $B_1 = N_Q$, in problem (4)-(5), where N_C and N_Q are the normal cone associated with C and Q, respectively. Then the SMIP becomes the classical split variational inequality problem (see [9, 10]). In addition, we get the SCNPP (see [8]) as a special case in problem (4)-(5) if we put $A = 0 = B$. As a result, it is clear that problems (4)-(5) are highly generic in nature and naturally contain a wide range of significant optimization issues, including split saddle-point problems, split equilibrium problems, split minimization problems, and split common fixed point problems. Moudafi [24], gave the following iterative technique

$$\begin{cases} x_1 \in H_1 \\ x_{n+1} = J_{\mu}^{A_1}(I^{H_1} - \mu A)(x_n + \gamma T^*(J_{\mu}^{B_1}(I^{H_2} - \mu B) - I^{H_2})T x_n), \end{cases} \tag{6}$$

where $\gamma \in (0, \frac{2}{\|T\|}), I^{H_1}, I^{H_2}$ are the identity operator on H_1 and H_2 respectively, and $J_\mu^{A_1}$ and $J_\mu^{B_1}$ are the re-solvents of A_1 and B_1, respectively. He established that the iterative sequence $\{x_n\}$ generated by Algorithm 6 converges weakly to a solution of (4)-(5) in as much the solution set of problem (4)-(5) is nonempty, A_1, B_1 are maximal monotone, and A, B are inverse-strongly monotone. Since the introduction of the SMIP, many authors have proposed and studied different iterative techniques to solve the SMIP (see [17, 20, 27, 32] and the references therein). However, all of these authors use the assumption that the operators A and B are inverse-strongly monotone, which may rule out some of the potential applications of these techniques.

Remark 1.1. It is therefore natural to ask, if an iterative technique can be developed with a weaker operator.

Izuchukwu et al., [19] provided an affirmative answer to the above remark by introducing the following iterative technique to solve the problem (4)-(5). In particular, they proposed the following iterative technique

Algorithm 1.2. Initialization Step: *Choose $x_0, x_1 \in H$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.*

Step 1: *Compute*

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n), \\ z_n &= Tw_n - \zeta\eta_n d_n \end{aligned}$$

where $d_n := Tw_n - y_n - \lambda_n(ATw_n - Ay_n)$, $\eta_n = \frac{\langle Tw_n - y_n, d_n}{\|d\|^2}$ if $d_n \neq 0$, otherwise, $\eta_n = 0$ and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu_1\|Tw_n - y_n\|}{\|ATw_n - Ay_n\|\}}, \lambda_n\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{7}$$

Step 2: *Compute*

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{8}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 3: *Compute*

$$\begin{aligned} u_n &= J_{\nu_n}^{B_1}(v_n - \nu_n Bv_n), \\ t_n &= v_n - \phi\omega_n b_n \end{aligned}$$

where $b_n = v_n - u_n - \nu_n(Bv_n - Bu_n)$, $\omega_n = \frac{\langle v_n - u_n, b\rangle}{\|b\|^2}$ and if $b_n \neq 0$, otherwise, $\omega_n = 0$ and

$$\nu_{n+1} = \begin{cases} \min\left\{\frac{\mu_2\|v_n - u_n\|}{\|Bv_n - Bu_n\|\}}, \nu_n\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \nu_n, & \text{otherwise.} \end{cases} \tag{9}$$

Step 4: *Compute*

$$x_{n+1} = (1 - \alpha_n)w_n + \alpha_n t_n, \tag{10}$$

where $J_{\lambda_n}^{A_1}$ and $J_{\nu}^{B_1}$ are the re-solvents of A_1 and B_1, respectively. They established that the iterative sequence $\{x_n\}$ generated by Algorithm 1.2 converges weakly to a solution of (4)-(5) in as much the solution set of problem (4)-(5) is nonempty, A_1 and B_1 are maximal monotone, A, B are monotone and Lipschitz continuous with Lipschitz constant L_1 and L_2, respectively. However, we observe the following regarding the iterative technique 1.2.

1. The authors established weak convergence. It is well known that strong convergence is more desirable in this area of research.
2. The authors claim that the value of the Lipschitz constant is not required. However, they used the fact that the operator is Lipschitz in their computation, thus, at some point the value of the Lipschitz constant might be needed (see).

In order to speed up the process of solving the smooth convex minimization problem, Polyak originally presented and examined the idea of inertial extrapolation in [25] in 1964. Since then, scientists have employed this method to accelerate the rate at which many iterative processes converge. Since its conception, the inertial extrapolation approach has been refined, extended, and generalized by numerous authors; see [1–5, 21, 27, 33, 34] and the references therein. Relaxation techniques have shown to be an effective method for improving the rate of convergence in this field of study. It's common knowledge that when inertial and relaxation techniques are combined, the results increase and the rate of convergence is higher than when either approach is used alone.

Motivated by the works of Moudafi [24], Shehu et. al., [27], Izuchukwu [19], Yao et. al., [32], Censor et al., [9] and the recent interest in this direction of research, our purpose in this study is to introduce and study a new inertial viscosity iterative technique for solving problem (4)-(5) in real Hilbert spaces. Our proposed iterative technique has the following properties.

1. The iterative sequence generated by our proposed iterative technique converges strongly, unlike the result obtained in [19].
2. Our proposed iterative technique does not require the strongly inversely monotone assumption on the operator A and B, which are used by authors in the literature (see [17,20,27,32] and the references therein). Instead, our proposed iterative technique requires that the operators A and B are to be monotone and Lipschitz continuous. In addition, the value of the Lipschitz constant is not relevant and the fact that the operator is Lipschitz is not used in our computation.
3. Our proposed iterative technique is made up of a new type of relaxed inertial technique, which helps speed the rate of convergence of the technique.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method and highlight some of its useful features. advantages over other existing algorithms. In Section 4, we establish strong convergence of our method and in Section 5, we applied the obtained result to the Split Equilibrium Problem (SEP). Lastly in Section 6, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces and in Section 7, we give the conclusion of the paper.

2. **Preliminaries.** In this section, we begin by recalling some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denotes strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{11}$$

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2. \tag{12}$$

$$\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y\rangle. \tag{13}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{14}$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{15}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then the operator T is called

1. L-Lipschitz continuous if there exists $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|,$$
for all $x, y \in H$. If $L = 1$, then T is called nonexpansive;
2. monotone if
$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H;$$
3. ?α-inversely strongly monotone if there exists $\alpha > 0$, such that
$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \ \forall \ x, y \in H.$$

If B is a multivalued operator, that is, $B : H \to 2^H$, then B is said to be monotone, if

$$\langle x - y, u - v\rangle \geq 0 \ \forall x, y \in H, \ u \in B(x), \ v \in B(y),$$

and B is maximal monotone, if the graph $G(B)$ of B defined by

$$G(B) := \{(x, y) \in H \times H : y \in B(x)\}$$

is not properly contained in the graph of any other monotone operator. It is generally known that B is maximal monotone if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(B)$ implies that $u \in B(x)$. Then the resolvent operator $J_\lambda^B : H \to H$ associated with B is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \ \forall \ x \in H,$$

where $\lambda > 0$ and I is the identity operator on H.
Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \min\{\|u - y\| \ \forall y \in C\}.$$

The normal cone of C at a point say $x \in H$ is given as

$$N_C = \{z \in H : \langle z, y - x\rangle \leq 0 \ \forall \ y \in C\}$$

if $x \in C$ and $\emptyset$ otherwise.

Lemma 2.2. *[26] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that ?*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{k\to\infty} a_n = 0$.

Lemma 2.3. *[22] Let H be a real Hilbert space, $A : H \to H$ be a monotone and Lipschitz continuous operator and $A_1 : H \to 2^H$ be maximal monotone operator, then, $(A + A_1) : H \to 2^H$ is maximal operator.*

3. Proposed Algorithm.

Assumption 3.1. Condition A. *Suppose*

1. *H_1 and H_2 are two real Hilbert spaces.*
2. *$B_1 : H_1 \to 2^{H_1}$ and $A_1 : H_2 \to 2^{H_2}$ be maximal monotone mappings and $T : H_1 \to H_2$ is a bounded linear operator with the adjoint operator T^*.*
3. *$B : H_1 \to H_1$ and $A : H_2 \to H_2$ are monotone and Lipschitz continuous with Lipschitz constant L_1 and L_2, respectively.*
4. *$f : H_1 \to H_1$ is a contraction mapping with $k \in [0,1)$.*
5. *The solution set of problem (4)-(5) is denoted $\Gamma \neq \emptyset$.*

Condition B. *Suppose that $\{\alpha_n\}$ is a real sequence such that*

1. *$\alpha_n \subset (0,1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.*
2. *$\{\epsilon_n\}$ is a positive integer such that $\circ(\alpha_n) = \epsilon_n$ means that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.*

We present the following iterative algorithm.

Algorithm 3.2. Initialization Step: *Given $\Gamma, \varphi > 0, \mu, \alpha, l, j \in (0,1), \phi, \zeta \in (0,2)$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.*

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{16}$$

Step 1: *Compute*

$$\begin{aligned} w_n &= (1-\alpha_n)x_n + (1-\alpha_n)\theta_n(x_n - x_{n-1}), \\ y_n &= J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n), \\ z_n &= Tw_n - \zeta\eta_n d_n \end{aligned}$$

where ?$d_n := Tw_n - y_n - \lambda_n(ATw_n - Ay_n)$, $\eta_n = (1-\mu)\frac{\|Tw_n - y_n\|^2}{\|d_n\|^2}$ if $d_n \neq 0$, otherwise $\eta_n = 0$ and λ_n is chosen to be the largest $\lambda \in \{\Gamma, \Gamma l, \Gamma l^2, \cdots\}$ satisfying

$$\lambda\|ATw_n - Ay_n\| \leq \mu\|Tw_n - y_n\|. \tag{17}$$

Step 2: *Compute*

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{18}$$

where γ_n *is chosen such that for small enough* $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ *if* $Tw_n \neq z_n$, *otherwise* $\gamma_n = \gamma$.

Step 3: *Compute*

$$\begin{aligned} u_n &= J^{B_1}_{\nu_n}(v_n - \nu_n B v_n), \\ t_n &= v_n - \phi\omega_n b_n \end{aligned}$$

where $b_n = v_n - u_n - \nu_n(Bv_n - Bu_n)$, $\omega_n = (1-\alpha)\frac{\|v_n - u_n\|^2}{\|b_n\|^2}$ *if* $b_n \neq 0$, *otherwise* $\omega_n = 0$ *and* ν_n *is chosen to be the largest* $\nu \in \{\varphi, \varphi j, \varphi j^2, \cdots\}$ *satisfying*

$$\nu\|Bv_n - Bu_n\| \leq \alpha\|v_n - u_n\|. \tag{19}$$

Step 4: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)t_n. \tag{20}$$

Remark 3.3. 1. The choice of the step size γ_n in Algorithms 3.2 do not require the prior knowledge of the operator norm $\|T||$. In addition, the step size is well defined. To see this, observe that

Proof. Let $p \in \Gamma$, observe that

$$\begin{aligned} \|z_n - Tp\|^2 &= \|Tw_n - \zeta\eta_n d_n - Tp\|^2 \\ &= \|Tw_n - Tp\|^2 + \|\zeta\eta_n d_n\|^2 - 2\zeta\eta_n\langle Tw_n - Tp, d_n\rangle. \end{aligned} \tag{21}$$

We now estimate $\langle Tw_n - Tp, d_n\rangle$. Since $p \in \Gamma$, then $0 \in (A_1 + A)Tp$ and since $y_n = J^{A_1}_{\lambda_n}(Tw_n - \lambda_n ATw_n)$, we have

$$Ay_n + \frac{1}{\lambda_n}(Tw_n - \lambda_n ATw_n - y_n) \in (A_1 + A)y_n \tag{22}$$

Using Lemma 2.3, we have $\langle y_n - Tp, Ay_n + \frac{1}{\lambda_n}(Tw_n - \lambda_n ATw_n - y_n)\rangle \geq 0$, as such, we have $\langle y_n - Tp, Tw_n - y_n - \lambda_n(ATw_n - Ay_n)\rangle \geq 0$.

$$\begin{aligned} &\langle Tw_n - Tp, d_n\rangle \\ &= \langle Tw_n - y_n + y_n - Tp, d_n\rangle \\ &= \langle Tw_n - y_n, d_n\rangle + \langle y_n - Tp, d_n\rangle \\ &= \langle Tw_n - y_n, Tw_n - y_n - \lambda_n(ATw_n - Ay_n)\rangle \\ &\quad + \langle y_n - Tp, Tw_n - y_n - \lambda_n(ATw_n - Ay_n)\rangle \\ &\geq (1-\mu)\|Tw_n - y_n\|^2 + \langle y_n - Tp, Tw_n - y_n - \lambda_n(ATw_n - Ay_n) \\ &\geq (1-\mu)\|Tw_n - y_n\|^2. \end{aligned} \tag{23}$$

Thus, using $\eta_n = (1-\mu)\frac{\|Tw_n - y_n\|^2}{\|d_n\|^2}$, we have (21)

$$\begin{aligned} \|z_n - Tp\|^2 &\leq \|Tw_n - Tp\|^2 + \|\zeta\eta_n d_n\|^2 - 2\zeta\eta_n(1-\mu)\|Tw_n - y_n\|^2 \\ &= \|Tw_n - Tp\|^2 + \|\zeta\eta_n d_n\|^2 - 2\zeta\|\eta_n d_n\|^2 \\ &= \|Tw_n - Tp\|^2 - \frac{(2-\zeta)}{\zeta}\|Tw_n - z_n\|^2 \\ &\leq \|Tw_n - Tp\|^2. \end{aligned} \tag{24}$$

Using the Cauchy-Schwarz inequality and (24), we have

$$\begin{aligned}\|T^*(Tw_n - z_n)\|\|w_n - p\| &\geq \langle T^*(Tw_n - z_n), w_n - p\rangle \\ &= \langle Tw_n - z_n, Tw_n - Tp\rangle \\ &= \frac{1}{2}[\|Tw_n - z_n\|^2 + \|Tw_n - Tp\|^2 - \|z_n - Tp\|^2] \\ &\geq \frac{1}{2}\|Tw_n - z_n\|^2. \end{aligned} \tag{25}$$

Since $z_n \neq Tw_n$, we have $\|Tw_n - z_n\| \geq 0$, thus, we obtain that

$$\|T^*(Tw_n - z_n)\|\|w_n - p\| > 0.$$

Hence, we have $\|T^*(Tw_n - z_n)\| \neq 0$ and so γ_n is well defined. □

2. We note that, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$. Clearly, we have that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it follows that
$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$
It is worth mentioning that, we can take $\alpha_n = 1/(n+1)^p$ and $\epsilon_n = 1/(n+1)^{1-p}$, where $p \in [0, 1/2)$.
3. It is well known that the Armijo-like search rule (17) and (19) are well defined.

4. **Convergence analysis.**

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2, under Assumption 3.1, we obtain that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$. By using the definition of w_n in Algorithm 3.2, we obtain

$$\begin{aligned}\|w_n - p\| &= \|(1-\alpha_n)x_n + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - p\| \\ &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\| + \alpha_n\|p\| \\ &= (1-\alpha_n)\|x_n - p\| + \alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\Big]. \end{aligned} \tag{26}$$

Using (65), we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\epsilon_n}{\alpha_n} \to 0$. Hence, we have

$$\lim_{n\to\infty}\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\Big] = \|p\|,$$

hence, there exists $N > 0$ such that $(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\| \leq N$. Thus, (26) becomes

$$\begin{aligned}\|w_n - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n N \\ &\leq \|x_n - p\| + \alpha_n N. \end{aligned} \tag{27}$$

In addition, using, (21), (23) and (24), we have

$$\|z_n - Tp\| \leq \|Tw_n - Tp\|. \tag{28}$$

Furthermore, using Algorithm 3.2 and the step size, we have

$$\begin{aligned}&\|v_n - p\|^2 \\ &= \|w_n + \gamma_n T^*(z_n - Tw_n) - p\|\end{aligned}$$

$$
\begin{aligned}
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 + 2\gamma_n \langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 + 2\gamma_n \langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 + \gamma_n \|z_n - Tp\|^2 - \gamma_n \|Tw_n - Tp\|^2 \\
&\quad - \gamma_n \|z_n - Tw_n\|^2 \\
&\le \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 + \gamma_n \|Tw_n - Tp\|^2 - \gamma_n \|Tw_n - Tp\|^2 \\
&\quad - \gamma_n \|z_n - Tw_n\|^2 \\
&\le \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 - \gamma_n(\gamma_n + \epsilon) \|T^*(z_n - Tw_n)\|^2 \\
&= \|w_n - p\|^2 - \gamma_n \epsilon \|T^*(z_n - Tw_n)\|^2 \le \|w_n - p\|^2, \qquad (29)
\end{aligned}
$$

which implies that

$$\|v_n - p\| \le \|w_n - p\|. \qquad (30)$$

Using similar approach as above, we have

$$\|t_n - p\|^2 \le \|v_n - p\|^2 - \frac{(2-\phi)}{\phi} \|v_n - t_n\|^2 \le \|v_n - p\|^2, \qquad (31)$$

which implies that

$$\|t_n - p\| \le \|v_n - p\|. \qquad (32)$$

Lastly, using Algorithm 3.2, (32), (30) and (27), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n) t_n\| \\
&\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|t_n - p\| \\
&\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|t_n - p\| \\
&\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|v_n - p\| \\
&\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|w_n - p\| \\
&\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)[\|x_n - p\| + \alpha_n N] \\
&\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n N \\
&= (1 - \alpha_n(1-k))\|x_n - p\| + \alpha_n (1-k) \left[\frac{N + \|f(p) - p\|}{(1-k)} \right] \\
&\le \max\left\{ \|x_n - p\|, \frac{N + \|f(p) - p\|}{(1-k)} \right\}. \qquad (33)
\end{aligned}
$$

It follows by induction

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{N + \|f(p) - p\|}{(1-k)} \right\}. \qquad (34)$$

Hence, $\{x_n\}$ is bounded. □

Theorem 4.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumption 3.1, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma \circ f(p)$.*

Let $p \in \Gamma$, using Algorithm 3.2, we have

$$
\begin{aligned}
&\|w_n - p\|^2 \\
&= \|x_n + \theta_n(x_n - x_{n-1}) - \alpha_n x_n - \theta_n \alpha_n (x_n - x_{n-1}) - p\|^2 \\
&= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1}) - \alpha_n p\|^2 \\
&\le \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(x_n - x_{n-1})\|^2 + 2\alpha_n \langle p, w_n - p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\theta_n(1-\alpha_n)\|x_n - p\|\|x_n - x_{n-1}\| + 2\alpha_n[\langle p, w_n - x_{n+1}\rangle + \langle p, x_{n+1} - p\rangle] \\
&\leq (1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\theta_n(1-\alpha_n)\|x_n - p\|\|x_n - x_{n-1}\| + 2\alpha_n[\langle p, w_n - x_{n+1}\rangle + \langle p, x_{n+1} - p\rangle] \\
&\leq (1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2\theta_n\|x_n - x_{n-1}\|[\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|] \\
&\quad + 2\alpha_n\|p\|\|w_n - x_{n+1}\| - 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M - 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\quad + 2\alpha_n\|p\|\|x_{n+1} - w_n\|, \qquad (35)
\end{aligned}
$$

where $M := sup_{n\in\mathbb{N}}\{\theta_n\|x_n - x_{n-1}\|, 2\|x_n - p\|\}$. In addition, using Algorithm 3.2, (32), (30), and (35), we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)t_n - p\|^2 \\
&= \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(t_n - p) + \alpha_n(f(p) - p)\|^2 \\
&\leq \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(t_n - p)\|^2 + 2\alpha_n\langle(f(p) - p), x_{n+1} - p\rangle \\
&\leq \alpha_n k^2\|x_n - p\|^2 + (1-\alpha_n)\|t_n - p\|^2 + 2\alpha_n\langle(f(p) - p), x_{n+1} - p\rangle \\
&\leq \alpha_n k\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n\langle(f(p) - p), x_{n+1} - p\rangle \\
&\leq \alpha_n k\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 + 2\alpha_n\langle(f(p) - p), x_{n+1} - p\rangle \\
&\leq \alpha_n k\|x_n - p\|^2 + (1-\alpha_n)[(1-\alpha_n)\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M \\
&\quad - 2\alpha_n\langle p, p - x_{n+1}\rangle + 2\alpha_n\|p\|\|x_{n+1} - w_n\|] + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n(1-k))\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M - 2(1-\alpha_n)\alpha_n\langle p, p - x_{n+1}\rangle \\
&\quad + 2\alpha_n\|p\|\|x_{n+1} - w_n\| + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&= (1-\alpha_n(1-k))\|x_n - p\|^2 + \alpha_n(1-k)\Bigg[\frac{\theta_n}{\alpha_n}\frac{1}{(1-k)}\|x_n - x_{n-1}\|M \\
&\quad - \frac{2(1-\alpha_n)}{(1-k)}\langle p, p - x_{n+1}\rangle + 2\frac{1}{(1-k)}\|p\|\|x_{n+1} - w_n\| \\
&\quad + 2\frac{1}{(1-k)}\langle f(p) - p, x_{n+1} - p\rangle\Bigg] \\
&= (1-\alpha_n(1-k))\|x_n - p\|^2 + \alpha_n(1-k)\Psi_n, \qquad (36)
\end{aligned}
$$

where, $\Psi_n = \frac{\theta_n}{\alpha_n}\frac{1}{(1-k)}\|x_n - x_{n-1}\|M - \frac{2(1-\alpha_n)}{(1-k)}\langle p, p - x_{n+1}\rangle + 2\frac{1}{(1-k)}\|p\|\|x_{n+1} - w_n\| + 2\frac{1}{(1-k)}\langle f(p) - p, x_{n+1} - p\rangle$. According to Lemma 2.2, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$
\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (37)
$$

To establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (37) holds. It is easy to see from (36) and (31) that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)t_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(t_n-p)+\alpha_n(f(p)-p)\|^2\\
&\le \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(t_n-p)\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)\|t_n-p\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)[\|v_n-p\|^2-\frac{(2-\phi)}{\phi}\|v_n-t_n\|^2]\\
&\quad +2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&= \alpha_n k\|x_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2-(1-\alpha_n)\frac{(2-\phi)}{\phi}\|v_n-t_n\|^2\\
&\quad +2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \|x_n-p\|^2+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M-2\alpha_n\langle p,p-x_{n+1}\rangle+2\alpha_n\|p\|\|x_{n+1}-w_n\|\\
&\quad -(1-\alpha_n)\frac{(2-\phi)}{\phi}\|v_n-t_n\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle, \qquad (38)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left((1-\alpha_{n_k})\frac{(2-\phi)}{\phi}\|v_{n_k}-t_{n_k}\|^2\right)\\
&\le \limsup_{k\to\infty}\Big[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|M-2\alpha_{n_k}\langle p,p-x_{n_k+1}\rangle\\
&\quad +2\alpha_{n_k}\|p\|\|x_{n_k+1}-w_{n_k}\|+2\alpha_{n_k}\langle (f(p)-p),x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Big]\\
&\le -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0.
\end{aligned}
$$

Thus, we have

$$
\lim_{k\to\infty}\|v_{n_k}-t_{n_k}\|=0. \qquad (39)
$$

In addition, using (36) and (29), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&= \|\alpha_n f(x_n)+(1-\alpha_n)t_n-p\|^2\\
&= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(t_n-p)+\alpha_n(f(p)-p)\|^2\\
&\le \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(t_n-p)\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)\|t_n-p\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)[\|w_n-p\|^2-\epsilon^2\|T^*(z_n-Tw_n)\|^2]\\
&\quad +2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \alpha_n k\|x_n-p\|^2+(1-\alpha_n)[(1-\alpha_n)\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|M\\
&\quad -2\alpha_n\langle p,p-x_{n+1}\rangle+2\alpha_n\|p\|\|x_{n+1}-w_n\|]-(1-\alpha_n)\epsilon^2\|T^*(z_n-Tw_n)\|^2\\
&\quad +2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle\\
&\le \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|M-2(1-\alpha_n)\alpha_n\langle p,p-x_{n+1}\rangle\\
&+2\alpha_n\|p\|\|x_{n+1}-w_n\|-(1-\alpha_n)\epsilon^2\|T^*(z_n-Tw_n)\|^2+2\alpha_n\langle (f(p)-p),x_{n+1}-p\rangle, \qquad (40)
\end{aligned}
$$

which implies that

$$\limsup_{k\to\infty}\left((1-\alpha_{n_k})\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2\right)$$
$$\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|M-2\alpha_{n_k}(1-\alpha_{n_k})\langle p,p-x_{n_k+1}\rangle$$
$$+2\alpha_{n_k}\|p\|\|x_{n_k+1}-w_{n_k}\|+2\alpha_{n_k}\langle(f(p)-p),x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Bigg]$$
$$\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.$$

Thus, we have

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \tag{41}$$

Using (25), we have

$$\|Tw_{n_k}-z_{n_k}\|\leq 2\|T^*(z_{n_k}-Tw_{n_k})\|\|w_{n_k}-p\|, \tag{42}$$

thus, using (41), we have

$$\lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}\|=0. \tag{43}$$

Furthermore, observe that

$$\begin{aligned}\|d_{n_k}\|&=\|Tw_{n_k}-y_{n_k}-\lambda_{n_k}(ATw_{n_k}-Ay_{n_k})\|\\&\leq\|Tw_{n_k}-y_{n_k}\|+\lambda_{n_k}\|ATw_{n_k}-Ay_{n_k})\|\\&\leq(1+\mu)\|Tw_{n_k}-y_{n_k}\|.\end{aligned} \tag{44}$$

In addition, using (44), we have

$$\eta_{n_k}=(1-\mu)\frac{\|Tw_{n_k}-y_{n_k}\|^2}{\|d_{n_k}\|^2}\geq(1-\mu)\frac{\|Tw_{n_k}-y_{n_k}\|^2}{(1+\mu)^2\|Tw_{n_k}-y_{n_k}\|^2}=\frac{(1-\mu)}{(1+\mu)^2}. \tag{45}$$

Thus, we have

$$\begin{aligned}\|Tw_{n_k}-y_{n_k}\|^2&=\frac{\eta_{n_k}}{(1-\mu)}\|d_{n_k}\|^2\\&=\frac{\|\zeta\eta_{n_k}d_{n_k}\|^2}{(1-\mu)\zeta^2\eta_{n_k}}=\frac{1}{(1-\mu)\zeta^2\eta_{n_k}}\|z_{n_k}-Tw_{n_k}\|^2\\&\leq\frac{1-\mu}{(1+\mu)^2\zeta^2}\|z_{n_k}-Tw_{n_k}\|^2,\end{aligned} \tag{46}$$

thus, using (43), we have

$$\lim_{k\to\infty}\|Tw_{n_k}-y_{n_k}\|=0. \tag{47}$$

Using a similar approach, we have

$$\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\|=0. \tag{48}$$

$$\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|\leq\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|+\lim_{k\to\infty}\|Tw_{n_k}-y_{n_k}\|=0. \tag{49}$$
$$\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|\leq\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|+\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=0. \tag{50}$$
$$\tag{51}$$

Also, using the above results, we have

$$\|t_{n_k}-u_{n_k}\|\leq\|t_{n_k}-v_{n_k}\|+\|v_{n_k}-u_{n_k}\|\to 0 \text{ as } k\to\ \infty, \tag{52}$$

$$\|z_{n_k} - y_{n_k}\| \le \|z_{n_k} - Tw_{n_k}\| + \|Tw_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{53}$$

In addition, we have that

$$\|w_{n_k} - x_{n_k}\| \le \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| + \alpha_{n_k}\|x_{n_k}\| + \alpha_{n_k}^2\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0$$
$$\text{as } k \to \infty. \tag{54}$$

Also, we have

$$\begin{aligned}
\|v_{n_k} - x_{n_k}\| &\le \|w_{n_k} - x_{n_k}\| + \gamma_{n_k}\|T^*(z_{n_k} - Tw_{n_k})\| \to 0 \text{ as } k \to \infty,\\
\|t_{n_k} - x_{n_k}\| &\le \|t_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty,\\
\|t_{n_k} - w_{n_k}\| &\le \|t_{n_k} - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| \to 0 \text{ as } k \to \infty,\\
\|y_{n_k} - x_{n_k}\| &\le \|y_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty,\\
\|z_{n_k} - x_{n_k}\| &\le \|z_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty,\\
\|x_{n_k+1} - t_{n_k}\| &\le \alpha_{n_k}\|f(x_{n_k}) - t_{n_k}\| \to 0 \text{ as } k \to \infty,\\
\|t_{n_k} - z_{n_k}\| &\le \|t_{n_k} - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Lastly, we have

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - t_{n_k}\| + \|t_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{55}$$

Now, since $\{x_{n_k}\}$ is bounded, then, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^* \in H_1$. Let $(v, u) \in G(A + A_1)$. Then $u - Av \in A_1 v$. Also it follow from (22), $\frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - \lambda_{n_{k_j}}ATw_{n_{k_j}} - y_{n_{k_j}}) \in Py_{n_{k_j}}$. Thus, using the monotonicity of A_1, we have

$$\langle v - y_{n_{k_j}}, u - Av - \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - \lambda_{n_{k_j}}ATw_{n_{k_j}} - y_{n_{k_j}})\rangle \ge 0 \tag{56}$$

Using (56) and the monotonicity of A, we have

$$\begin{aligned}
&\langle v - y_{n_{k_j}}, u\rangle\\
&\ge \langle v - y_{n_{k_j}}, Av + \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}}) - ATw_{n_{k_j}}\rangle\\
&= \langle v - y_{n_{k_j}}, Av - Ay_{n_{k_j}}\rangle + \langle v - y_{n_{k_j}}, Ay_{n_{k_j}} - ATw_{n_{k_j}}\rangle\\
&\quad + \langle v - y_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}})\rangle\\
&\ge \langle v - y_{n_{k_j}}, Ay_{n_{k_j}} - ATw_{n_{k_j}}\rangle + \langle v - y_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}})\rangle.
\end{aligned} \tag{57}$$

From (54), we can choose a subsequence $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$ such that $\{w_{n_{k_j}}\}$ converges weakly to x^*. Also, since T is a bounded linear operator, we have that $\{Tw_{n_{k_j}}\}$ converges weakly to Tx^*. Hence, using (47), we have that $\{y_{n_{k_j}}\}$ converges weakly to Tx^*. Using (17) and (47), we have

$$\lim_{j\to\infty}\|ATw_{n_{k_j}} - Ay_{n_{k_j}}\| \le \frac{\mu}{\lambda}\lim_{j\to\infty}\|Tw_{n_{k_j}} - y_{n_{k_j}}\| = 0.$$

Thus, we have that

$$\langle v - Tx^*, u\rangle \ge 0$$

as $j \to \infty$. Thus, by the maximal monoticity of $A+A_1$, we have $Tx^* \in (A+A_1)^{-1}(0)$. Also, using similar approach as above, (41) and (48), we have

$$\lim_{j\to\infty} \|Bv_{n_{k_j}} - Bu_{n_{k_j}}\| = 0$$

and

$$\lim_{j\to\infty} \|v_n - w_n\| = \lim_{j\to\infty} \gamma_{n_{k_j}} \|T^*(z_{n_{k_j}} - Tw_{n_{k_j}})\| = 0.$$

Thus, since $w_{n_{k_j}}, v_{n_{k_j}}$ and $u_{n_{k_j}}$ converges weakly to x^* and

$$\lim_{j\to\infty} \|Bv_{n_{k_j}} - Bu_{n_{k_j}}\| = 0,$$

we have $x^* \in (B + B_1)^{-1}(0)$. Thus, $x^* \in \Gamma$. Furthermore, since $x_{n_{k_j}}$ converges weakly to x^*, we obtain

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty} \langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, x^* - p\rangle. \quad (58)$$

Since p is a solution of Γ, it follows

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, x^* - p\rangle \leq 0, \quad (59)$$

and we obtain from (59) and (55)

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \leq 0. \quad (60)$$

Lastly, we also obtain

$$\|x_{n_k+1} - w_{n_k}\| \leq \|x_{n_k+1} - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| \to 0 \text{ as } n \to \infty. \quad (61)$$

Using our assumption, (61), (60), and the fact that $\Psi_{n_{k_j}} = \frac{\theta_{n_{k_j}}}{\alpha_{n_{k_j}}}\frac{1}{(1-k)}\|x_{n_{k_j}} - x_{n_{k_j}-1}\|M - \frac{2(1-\alpha_{n_{k_j}})}{(1-k)}\langle p, p - x_{n_{k_j}+1}\rangle + 2\frac{1}{(1-k)}\|p\|\|x_{n_{k_j}+1} - w_n\| + 2\frac{1}{(1-k)}\langle f(p) - p, x_{n_{k_j}+1} - p\rangle \leq 0$.Thus, From Lemma 2.2, we have that $\lim_{n\to\infty} \|x_n - p\| = 0$.

5. **Application to split equilibrium problem.** An interesting optimization problem is the equilibrium problem (EP) introduced and studied by Blum and Oettli [7]. Some well known problems in sciences are special type of the equilibrium problem. For example, Minimization problems, mathematical programming problems, saddle point problems, Nash equilibrium problems fixed point problems, vector minimization problems, and so on. The equilibrium problem (EP) is defined as finding $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad (62)$$

for all $y \in C$, where $F : C\times C \to \mathbb{R}$ is a bifunction. The solution set of EP is denoted by $EP(F)$. Due to its numerous application, the notion of EP has been extended and generalized by different scholars. For instant, Kazmi and Rizvi [20] introduced and studied the following Split Equilibrium Problem (SEP). Let $C \subseteq H_1, Q \subseteq H_2$, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and suppose that $T : H_1 \to H_2$ is a bounded linear operator. The SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0 \ \forall \ x \in C \quad (63)$$

and such that

$$y^* = Tx^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \ \forall y \in Q. \quad (64)$$

We denote the solution set of (63) and (64) by $\Omega = \{x \in EP(F_1) : Tx \in EP(F_2)\}$. It is easy to see that if $H_1 = H_2 = H$ and $F_2 = 0$ with $T = I_H$ problem (63)-(64) becomes (62). The following assumptions are needed in solving equilibrium problems.

Assumption 5.1. *[30] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

1. *$F(x,x) = 0$ for all $x \in C$;*
2. *F is monotone. That is $F(x,y) + F(y,x) \leq 0$ for all $x \in C$;*
3. *for each $x, y, z \in C, \limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F_1(x,y)$;*
4. *for each $x \in C, y \mapsto F_1(x,y)$ is convex and weakly lower semi-continuous.*

Lemma 5.2. *[30] Assume that $F : C \times C \to \mathbb{R}$ satisfies Assumption 5.1 and let B_F be a set valued operator defined from H into itself as*

$$B_F(x) = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z\rangle \ \forall y \in C\}, & \text{if } x \in C, \\ \\ \emptyset, & \text{otherwise.} \end{cases} \tag{65}$$

Then B_F is a maximal monotone operator with domain $D(B_F) \subset C$ and $B^{-1}(0) = EP(F)$. Thus, the re-solvent $J_\lambda^F := (I_H + \lambda B_F)^{-1}$ of B_F is defined by

$$J_\lambda^F(x) = \{x \in C : F(z,y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0 \ \forall \ y \in C\}.$$

Applying of Lemma 5.2, the following iterative method is deduced from Algorithm 3.2 for solving the problem (63)-(64).

Algorithm 5.3. Initialization Step: *Given $\lambda, \gamma, \mu > 0$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.*

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{66}$$

Step 1: *Compute*

$$\begin{aligned} w_n &= (1-\alpha_n)x_n + (1-\alpha_n)\theta_n(x_n - x_{n-1}), \\ y_n &= J_\lambda^{F_2} T w_n. \end{aligned}$$

Step 2: *Compute*

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n). \tag{67}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$. **Step 3:** *Compute*

$$u_n = J_\mu^{F_1} v_n.$$

Step 4: *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n. \tag{68}$$

Theorem 5.4. *Let $F_1 : H_1 \times H_1 \to \mathbb{R}$ and $F_2 : H_2 \times H_2 \to \mathbb{R}$ be two bifunctions which satisfy conditions (5.1) and suppose that $\Omega \neq \emptyset$. Also, let Assumptions 3.1 hold. Then, the sequence $\{x_n\}$ generated by Algorithm 5.3 converges strongly to an element in Ω.*

6. **Numerical example.** In this section, we will give some numerical examples which will show the applicability and the efficiency of our proposed iterative method in comparison to the same Algorithm 3.2 but with the normal inertial $(x_n+\theta_n(x_n-x_{n-1}))$ and Algorithm 3.2 without inertial term.

Example 6.1. *Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \forall\ x, y \in L_2([0,1]) \ \textit{and} \ \|x\|^2 := \int_0^1 |x(t)|^2 dt \ \forall x, y, \in L_2([0,1]).$$

Let $T; A; A_1; B; B_1; f : L_2([0,1]) \to L_2([0,1])$ be defined by;

$$\begin{aligned} A_1x(t) &= \max\{0, x(t)\},\ t \in [0,1], x \in L_2([0,1]);\\ B_1x(t) &= \frac{1}{2}\max\{0, x(t)\},\ t \in [0,1], x \in L_2([0,1]);\\ Ax(t) &= \int_0^t x(s)ds,\ t \in [0,1], x \in L_2([0,1]),\\ Bx(t) &= \int_0^1 \left(x(t) - \frac{2tse^{t+s}}{e\sqrt{e^2-1}} cosx(s)\right)ds,\ t \in [0,1], x \in L_2([0,1]),\\ fx(t) &= \int_0^t \frac{t}{2}x(s)\ ds\ t \in [0,1], x \in L_2([0,1]);\\ Tx(s) &= \int_0^1 K(s,t)x(t)dt\ x \in L_2([0,1]), \end{aligned}$$

where K is a continuous real-valued function on $[0,1] \times [0,1]$. It is easy to see that A, A_1, B, B_1 and T satisfies Assumption 3.1, (see [6, 18] for details). In addition, f is a contraction on $L_2([0,1])$ and T is a bounded linear operator with the adjoint operator $T^*x(s) = \int_0^1 K(t,s)x(t)dt\ x \in L_2([0,1])$. We choose $\alpha_n = \frac{2}{200n+5}, \Gamma = 0.4, \varphi = 0.3, l = 0.5, j = 0.4, \mu = 0.2, \alpha = 0.3, \zeta = 1.6$ and ϕ for all $n \in \mathbb{N}$. Also if we consider $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case I: $x_0(t) = t + 2,\ x_1(t) = 2t + 1;$
Case II: $x_0(t) = e^{2t} + 1,\ x_1(t) = t^3;$
Case III: $x_0(t) = t^3 + t^2 + 2,\ x_1(t) = 2t + \cos(t).$

Example 6.2. *Let $H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Suppose the operators $T; A; A_1; B; B_1;$ $f : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by*

$$\begin{aligned} A_1x &= (3x_1, 3x_2, 3x_3, \cdots, 3x_i, \cdots)\ \forall\ x \in l_2(\mathbb{R});\\ B_1x &= (7x_1, 7x_2, 7x_3, \cdots, x7_i, \cdots)\ \forall\ x \in l_2(\mathbb{R});\\ A(x) &= 2(x_1, x_2, x_3, \cdots, x_i, \cdots)\ \forall\ x \in l_2(\mathbb{R});\\ B(x) &= (\frac{x_1+|x_1|}{3}, \frac{x_2+|x_2|}{3}, \frac{x_3+|x_3|}{3}, \cdots, \frac{x_i+|x_i|}{3}, \cdots)\ \forall\ x \in l_2(\mathbb{R});\\ Tx &= (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ x \in l_2(\mathbb{R});\\ f(x) &= \frac{x}{3}\ \forall\ x \in l_2(\mathbb{R}). \end{aligned}$$

It is easy to see that T is a bounded linear operator with the adjoint operator $T^*y = (0, y_1, \frac{y_2}{2}, \frac{y_3}{3}, \cdots)\ y \in l_2(\mathbb{R})$ and A, A_1, B, B_1 satisfy Assumptions 3.1. We choose

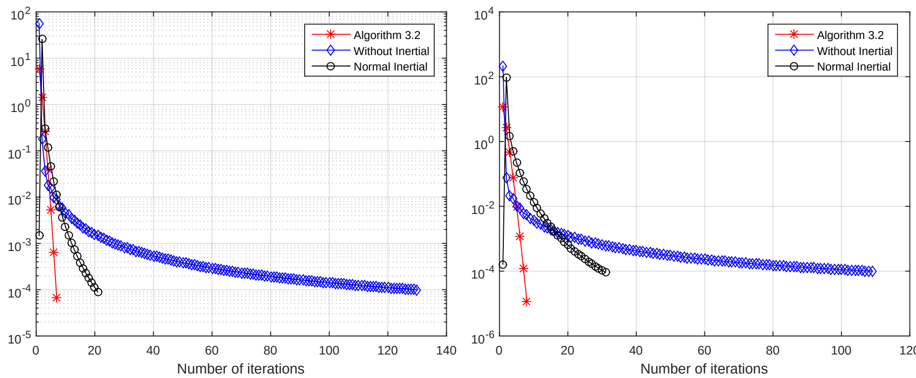

FIGURE 1. Example 6.1, Top Left: Case I; Top Right: Case II.

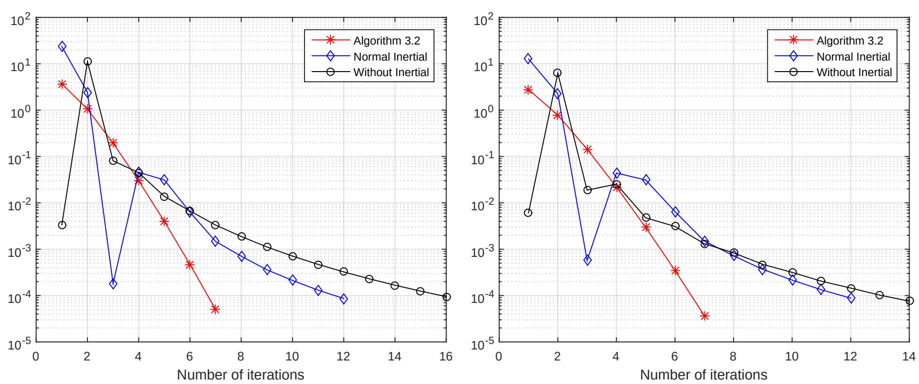

FIGURE 2. Example 6.2, Top Left: Case I; Top Right II.

$\alpha_n = \frac{2}{200n+5}, \Gamma = 0.4, \varphi = 0.3, l = 0.5, j = 0.4, \mu = 0.2, \alpha = 0.3, \zeta = 1.6$ *and* ϕ *for all* $n \in \mathbb{N}$. *Also if we consider* $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ *as the stopping criterion and choose the following as starting points:*

Case I: $x_0 = (1, 3, 5, \cdots), \; x_1 = (0.5, 0.5, 0.5, \cdots)$;
Case II: $x_0 = (1, 2, 3, 4, \cdots), \; x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \cdots)$.

7. **Conclusion.** In this paper we have introduced and studied an iterative algorithm for solving split monotone inclusions problem in the framework of real Hilbert spaces using viscosity inertial techniques. We have obtained a strong convergence result without assuming that our single valued operator is inversely strongly monotone assumption. We emphasize that the value of the Lipschitz constant is not required for the iterative technique to be implemented, and during computation,

the Lipschitz continuity was not used. Lastly, we present an application and also some numerical experiments to show the efficiency and the applicability of our proposed iterative method. However, the linear rate of convergence of the iterative technique introduced and studied in this paper was not investigated. Hence, we intend to look in this direction in the near future.

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Declarations. The authors declare that they have no conflict of interest. In addition, we never used any form of artificial intelligence in the writing of this paper.

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