Asia Pac. J. Math. 2024 **11**:84

A NEW RELAXED INERTIAL ISHIKAWA-TYPE ALGORITHM FOR SOLVING FIXED POINTS PROBLEMS WITH APPLICATIONS TO CONVEX OPTIMIZATION PROBLEMS

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Received Jul. 22, 2024

Abstract. In this research, we present a new relaxed intertial algorithm without viscosity for solving common solution of countable family of nonexpansive mappings in real Hilbert spaces. We obtain the strong convergence results of the proposed method under some wild conditions on the control parameters. We apply our main results to solve convex bilevel optimization problems. Finally, we present a numerical example to illustrate the efficiency of our method over some existing methods in the literature.

2020 Mathematics Subject Classification. 47H06; 47H09; 47J05; 47J25.

Key words and phrases. fixed point; inertial technique; convex optimization problem; Hilbert space.

1. Introduction

Fixed point theory plays very essential role in the fields of pure and applied mathematics. It has several applications in engineering and applied sciences. For instance, fixed point theory can be used to solve signal processing problem, image restoration problem, optimal control problem, equilibrium problem, variational inequality problem, game theory, among others.

Let C be a nonempty subset of a real Hilbert space H. Then, a point $x \in C$ is called a fixed point of a mapping $S : C \to C$ if, $Sx = x$. We denote the set of the fixed points of S by $F(S) = \{x \in C : Tx = x\}$. The mapping S is said to be nonexpansive if, for each $x, y \in C$, we have $\|Sx - Sy\| \leq \|x - y\|$. S is said to be quasinonexpansive if $F(S) \neq \emptyset$, then $\|Sx - p\| \leq \|x - p\|, \forall p \in F(S), x \in C$.

Many methods have introduced by different authors for approximating the fixed points of various mappings [1,2,4,5,16–20]. For $\{\alpha_m\}, \{\beta_m\}$ and $\{\gamma_m\} \in (0, 1)$, the Mann [2], Ishikawa [1] and Noor [3] are given as follows:

DOI: 10.28924/APJM/11-84

$$\begin{cases} x_0 \in C, \\ x_{m+1} = (1-\alpha_m)x_m + \alpha_m S x_m. \end{cases} \tag{1}$$

$$\begin{cases} x_0 \in C, \\ z_m = (1-\beta_m)x_m + \beta_m S x_m, \\ x_{m+1} = (1-\alpha_m)x_m + \alpha_m S z_m. \end{cases} \tag{2}$$

$$\begin{cases} x_0 \in C, \\ w_m = (1-\gamma_m)x_m + \gamma_m S x_m, \\ z_m = (1-\beta_m)x_m + \beta_m S w_m, \\ x_{m+1} = (1-\alpha_m)x_m + \alpha_m S z_m. \end{cases} \tag{3}$$

It is not hard to see that Ishikawa method is a two two step Mann method and the Noor method is a three step Mann method.

The inertial technique has been used widely by many authors in recent years to enhance the speed of convergence of iterative methods for solving fixed point problems and optimization problems [6–15]. In [9], Jailoka et al. introduced an inertial viscosity method for finding the common fixed point of a countable family nonexpansive mapping. The authors proved the strong convergence results of their methods in real Hilbert spaces. Recently, Janngam et al. [15] introduced an inertial viscosity SP algorithm for solving the convex bilevel optimization problem in real Hilbert space.

It is worth mentioning that the strong converge results of the above methods reply on the viscosity technique and this make the computation of this methods more complex.

Motivated by the above results, in this research, we present a new relaxed intertial algorithm without viscosity for solving common solution of countable family of nonexpansive mappings in real Hilbert spaces. We obtain the strong convergence results of the proposed method under some wild conditions on the control parameters. We apply our main results to solve convex bilevel optimization problems. Finally, we present a numeral example to illustrate the efficiency of our method over some existing methods in the literature.

2. Preliminaries

In this section, we give some lemmas that will be useful in obtaining our strong convergence results.

Lemma 2.1. *A sequence $\{S_m\}$ with $\cap_{m=1}^{\infty} F(S_m) \neq \emptyset$ is said to be satisfy the condition (Z) if for every bounded sequence $\{x_m\}$ in C such that*

$$\lim_{m\to\infty} \|x_m - S_m x_m\| = 0,$$

then, every weak cluster point of $\{x_m\}$ *belongs to* $\cap_{m=1}^{\infty} F(S_m) \neq \emptyset$.

For every point $u \in \mathcal{H}$, the unique nearest point which is denoted by $P_C u$ exists in C such that $\|u - P_C u\| \leq \|u - v\|$, $\forall v \in C$. The mapping P_C is called the metric projection of $\mathcal{H}$ onto C and it is known to be nonexpansive.

Lemma 2.2. [29] *Let* $\mathcal{H}$ *be a real Hilbert space and* C *a nonempty closed convex subset of* H. *Suppose* $u \in \mathcal{H}$ *and* $v \in C$. *Then* $v = P_C u \iff \langle u - v, v - w\rangle \geq 0$, $\forall w \in C$.

Lemma 2.3. *Let* H *be a real Hilbert space. Then for every* $u, v \in H$ *and* $\sigma \in \mathbb{R}$, *we have*

(i) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$;

(ii) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$;

(iii) $\|\sigma u + (1-\sigma)v\|^2 = \sigma\|u\|^2 + (1-\sigma)\|v\|^2 - \sigma(1-\sigma)\|u - v\|^2$.

Lemma 2.4. [28] *Let* $\{u_k\}$ *be a sequence of non-negative real numbers such that*

$$a_{k+1} \leq (1 - \nu_k)a_k + \nu_k b_k, \ \forall k \geq 1,$$

where $\{\nu_k\} \subset (0, 1)$ *with* $\sum_{k=0}^{\infty} \nu_k = \infty$. *If* $\limsup\limits_{k\to\infty} b_k \leq 0$ *for every subsequence* $\{a_{k_j}\}$ *of* $\{a_k\}$, *the following inequality hold:*

$$\liminf_{k\to\infty}(a_{k_{j+1}} - a_{k_j}) \geq 0,$$

Then $\lim\limits_{k\to\infty} a_k = 0$.

3. Propose Algorithm

In this section, we give the our suggested algorithm and outline some of its properties. Firstly, the strong convergence theorem for the algorithm will be obtained under the following assumptions:

Assumption 3.1. Conditions on the various operators and control parameters.

(A_1) Let C be a subset of a real Hilbert space H.

(A_2) $S_m : C \to C$ is a nonexpasive mapping.

(A_3) $\{\gamma_m\} \subset [a, b] \subset (0, 1]$ and $\{\beta_m\} \subset [c, d] \subset (0, 1]$.

(A_4) $\{\alpha_m\} \subset (0, 1)$ such that $\lim_{m\to\infty} \alpha_m = 0$, $\sum_{m=1}^{\infty} \alpha_m = \infty$.

(A_5) The positive sequence $\{\varepsilon_m\}$ satisfies $\lim_{m\to\infty} \frac{\varepsilon_m}{\alpha_m} = 0$.

Below is the proposed method.

Algorithm 3.2. Relax inertial Ishikawa-type algorithm.
Initialization: Choose $\psi > 0$, $x_0, x_1 \in \mathcal{H}_1$, and set $m = 1$.

Iterative Steps: Calculate the next iteration point x_{m+1} as follows:

Step 1: Choose ψ_m such that $\psi_m \in [0, \bar{\psi}_m]$, where

$$\bar{\psi}_m = \begin{cases} \min\left\{\psi, \frac{\varepsilon_m}{\|x_m - x_{m-1}\|}\right\}, & \text{if } \ x_m \neq x_{m-1}, \\ \psi, & \text{otherwise.} \end{cases} \tag{4}$$

Step 2: Set

$$w_m = (1-\alpha)(a_m + \psi_m(a_m - a_{m-1})) \tag{5}$$

and compute

$$z_m = (1-\beta_m)w_m + \beta_m S_m w_m. \tag{6}$$

Step 3: Compute

$$x_{m+1} = (1-\gamma_m)z_m + \gamma_m S_m z_m. \tag{7}$$

Put $m := m + 1$ and return to **Step 1**.

---

4. Convergence Analysis

Theorem 4.1. *Suppose $\{x_m\}$ is the sequence generated by algorithm 3.2 under Assumption 3.1. Then, $\{x_m\}$ converges strongly to an element $x^\dagger \in \Omega$, where $\|x^\dagger\| = \min\{\|p^\dagger\|, p^\dagger \in \Omega\}$.*

Proof. we will divide the proof into four claims as follows:

Claim 1: The sequence $\{x_m\}$ is bounded. Indeed, let $x^\dagger \in \Omega$. From $\|x^\dagger\| = \min\{\|p^\dagger\| : p^\dagger \in \Omega\}$, we obtain $p^\dagger = P_\Omega(0)$. Then from From (4), we have $\psi_m\|x_m - x_{m-1}\| \leq \varepsilon_m$, $\forall m \in \mathbb{N}$. Now, since by Assumption 3.1 (A_5) we have that $\lim_{m\to\infty} \frac{\varepsilon_m}{\alpha_m} = 0$. This implies that

$$\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| \leq \frac{\varepsilon_m}{\alpha_m} \to 0, \ \text{as}\, m \to \infty. \tag{8}$$

Using (5), we have

$$\begin{aligned} \|w_m - x^\dagger\| &= \|x_m + \psi_m(x_m - x_{m-1}) - \alpha_m[x_m + \psi_m(x_m - x_{m-1})] - x^\dagger\| \\ &= \|(1-\alpha_k)(x_m - x^\dagger) + (1-\alpha_m)\psi_m(x_m - x_{m-1}) - \alpha_m x^\dagger\| \\ &\leq \|(1-\alpha_k)\|x_m - x^\dagger\| + (1-\alpha_m)\psi_m\|x_m - x_{m-1}\| + \alpha_m\|x^\dagger\| \\ &= (1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m\left[(1-\alpha_m)\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| + \|x^\dagger\|\right]. \end{aligned} \tag{9}$$

By (8), we have

$$\lim_{m\to\infty}\left[(1-\alpha_m)\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| + \|x^\dagger\|\right] = \|x^\dagger\|. \tag{10}$$

Therefore, a constant $C_1 > 0$ exists such that

$$(1-\alpha_m)\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| + \|x^\dagger\| \le C_1, \ \forall m \in \mathbb{N}. \tag{11}$$

From (9) and (11), we obtain

$$\|w_m - x^\dagger\| \le (1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m C_1, \ \forall m \in \mathbb{N}. \tag{12}$$

From (6), we have

$$\begin{aligned}
\|z_m - x^\dagger\| &= \|(1-\beta_m)w_m + \beta_m S_m w_m - x^\dagger\| \\
&\le (1-\beta_m)\|w_m - x^\dagger\| + \beta_m\|S_m w_m - x^\dagger\| \\
&\le (1-\beta_m)\|w_m - x^\dagger\| + \beta_m\|w_m - x^\dagger\| \\
&= \|w_m - x^\dagger\|.
\end{aligned} \tag{13}$$

Also, by (7), we have

$$\begin{aligned}
\|x_{m+1} - x^\dagger\| &= \|(1-\gamma_m)z_m + \gamma_m S_m z_m - x^\dagger\| \\
&\le (1-\gamma_m)\|z_m - x^\dagger\| + \gamma_m\|S_m z_m - x^\dagger\| \\
&\le (1-\gamma_m)\|z_m - x^\dagger\| + \gamma_m\|z_m - x^\dagger\| \\
&= \|z_m - x^\dagger\|.
\end{aligned} \tag{14}$$

Combing (12), (13) and (14), we obtain

$$\begin{aligned}
\|x_{m+1} - x^\dagger\| &= (1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m C_1 \\
&\le \max\{\|x_m - x^\dagger\|, C_1\} \\
&\vdots \\
&\le \max\{\|x_{m_0} - x^\dagger\|, C_1\},
\end{aligned}$$

Therefore, the sequence $\{x_m\}$ is bounded. Consequently, $\{z_m\}$, $\{w_m\}$, $\{S_m z_m\}$ and $\{S_m z_m\}$ are bounded sequences.

Claim 2: The following inequality holds for some $C_2 > 0$:

$$\beta_m(1-\beta_m)\|w_m - S_m w_m\|^2 + \gamma_m(1-\gamma_m)\|z_m - S_m z_m\|^2 \le \|x_m - x^\dagger\|^2 - \|x_{m+1} - x^\dagger\|^2 + \alpha_m C_2. \tag{15}$$

Indeed from, (12), we have

$$\begin{aligned}\|w_m - x^\dagger\|^2 &\le (1-\alpha_m)^2\|x_m - x^\dagger\|^2 + 2\alpha_m C_1(1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m^2 C_1^2 \\ &\le (1-\alpha_m)\|x_m - x^\dagger\|^2 + \alpha_m[2C_1(1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m C_1^2] \\ &\le \|x_m - x^\dagger\|^2 + \alpha_m C_2, \end{aligned} \tag{16}$$

where $C_2 = \max\{2C_1(1-\alpha_m)\|x_m - x^\dagger\| + \alpha_m C_1^2 : m \in \mathbb{N}\}$.

Now, from (6) and Lemma (2.3), we have

$$\begin{aligned}\|z_m - x^\dagger\|^2 &= (1-\beta_m)\|w_m - x^\dagger\|^2 + \beta_m\|S_m w_m - x^\dagger\|^2 - \beta_m(1-\beta_m)\|w_m - S_m w_m\|^2 \\ &\le (1-\beta_m)\|w_m - x^\dagger\|^2 + \beta_m\|w_m - x^\dagger\|^2 - \beta_m(1-\beta_m)\|w_m - S_m w_m\|^2 \\ &= \|w_m - x^\dagger\|^2 - \beta_m(1-\beta_m)\|w_m - S_m w_m\|^2. \end{aligned} \tag{17}$$

Also, from (7) and Lemma (2.3), we have

$$\begin{aligned}\|x_{m+1} - x^\dagger\|^2 &= (1-\gamma_m)\|z_m - x^\dagger\|^2 + \gamma_m\|S_m z_m - x^\dagger\|^2 - \gamma_m(1-\gamma_m)\|z_m - S_m z_m\|^2 \\ &\le (1-\gamma_m)\|z_m - x^\dagger\|^2 + \gamma_m\|z_m - x^\dagger\|^2 - \beta_m(1-\gamma_m)\|z_m - S_m z_m\|^2 \\ &= \|z_m - x^\dagger\|^2 - \beta_m(1-\gamma_m)\|z_m - S_m z_m\|^2. \end{aligned} \tag{18}$$

Combing (16), (17) and (18), we have

$$\|x_{m+1} - x^\dagger\|^2 \le \|x_m - x^\dagger\|^2 - \beta_m(1-\beta_m)\|w_m - S_m w_m\|^2 - \gamma_m(1-\gamma_m)\|z_m - S_m z_m\|^2 + \alpha_m C_2.$$

This implies that

$$\beta_m(1-\beta_m)\|w_m - S_m w_m\|^2 + \gamma_m(1-\gamma_m)\|z_m - S_m z_m\|^2 \le \|x_m - x^\dagger\|^2 - \|x_{m+1} - x^\dagger\|^2 + \alpha_m C_2.$$

Claim 3: The following inequality holds for all $m \in \mathbb{N}$:

$$\begin{aligned}\|x_{m+1} - x^\dagger\|^2 &\le (1-\alpha_m)\|x_m - x^\dagger\|^2 + 2(1-\alpha_m)\psi_m\|x_m - x^\dagger\|\|x_m - x_{m-1}\| \\ &\quad +\psi_m^2\|x_m - x_{m-1}\|^2 + 2\alpha_m\langle -x^\dagger, w_m - x^\dagger\rangle \end{aligned} \tag{19}$$

$$\begin{aligned} &= (1-\alpha_m)\|x_m - x^\dagger\|^2 \\ &\quad +\alpha_m \left\{ \begin{array}{c} 2(1-\alpha_m)\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\|\|x_m - x^\dagger\| \\ +\psi_m\|x_m - x_{m-1}\|\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| \\ +2\|x^\dagger\|\|w_m - x_{m+1}\| + 2\langle -x^\dagger, x_{m+1} - x^\dagger\rangle \end{array} \right\}. \end{aligned} \tag{20}$$

Indeed, from Lemma 2.3, (5), (17) and (18), obtain

$$\begin{aligned}\|x_{m+1} - x^\dagger\|^2 &\le \|w_m - x^\dagger\|^2 \\ &= \|(1-\alpha_m)(x_m - x^\dagger) + (1-\alpha_m)\psi_m(x_m - x_{m-1}) - \alpha_m x^\dagger\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \|(1-\alpha_k)(x_m - x^\dagger) + (1-\alpha_m)\psi_m(x_m - x_{m-1})\|^2 + 2\alpha_m\langle -x^\dagger, w_m - x^\dagger\rangle \\
&\leq (1-\alpha_m)^2\|x_m - x^\dagger\|^2 + 2(1-\alpha_m)^2\psi_m\|x_m - x^\dagger\|\|x_m - x_{m-1}\| \\
&\quad +(1-\alpha_m)^2\psi_m^2\|x_m - x_{m-1}\|^2 + 2\alpha_m\langle -x^\dagger, w_m - x^\dagger\rangle \\
&\leq (1-\alpha_m)\|x_m - x^\dagger\|^2 + 2(1-\alpha_m)\psi_m\|x_m - x^\dagger\|\|x_m - x_{m-1}\| \\
&\quad +\psi_m^2\|x_m - x_{m-1}\|^2 + 2\alpha_m\langle -x^\dagger, w_m - x^\dagger\rangle \\
&= (1-\alpha_m)\|x_m - x^\dagger\|^2 \\
&\quad +\alpha_m\left\{\begin{array}{c} 2(1-\alpha_m)\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\|\|x_m - x^\dagger\| \\ +\psi_m\|x_m - x_{m-1}\|\frac{\psi_m}{\alpha_m}\|x_m - x_{m-1}\| \\ +2\|x^\dagger\|\|w_m - x_{m+1}\| + 2\langle -x^\dagger, x_{m+1} - x^\dagger\rangle \end{array}\right\}. \qquad (21)
\end{aligned}
$$

Claim 4: The sequence $\{\|x_m - x^\dagger\|^2\}$ converges to zero.

Indeed, from Lemma 2.4, (20) and (8), it suffices to prove that $\limsup_{j\to\infty}\langle -x^\dagger, x_{m_j+1} - x^\dagger\rangle \leq 0$ and $\lim_{j\to\infty}\|w_{m_j} - x_{m_j+1}\| = 0$, for every subsequence $\{\|x_{m_j} - x^\dagger\|\}$ of $\{\|x_m - x^\dagger\|\}$ fulfilling

$$
\liminf_{j\to\infty}(\|x_{m_j+1} - x^\dagger\| - \|x_{m_j} - x^\dagger\|) \geq 0. \qquad (22)
$$

In what follows, suppose $\{\|x_{m_j} - x^\dagger\|\}$ is a subsequence of $\{\|x_m - x^\dagger\|\}$ such that (22) holds. Then, from (15) and Assumption 3.1, we have

$$
\begin{aligned}
&\limsup_{j\to\infty}\left[\beta_m(1-\beta_m)\|w_{m_j} - S_{m_j}w_{m_j}\|^2 + \gamma_m(1-\gamma_m)\|z_{m_j} - S_{m_j}z_{m_j}\|^2\right] \\
&\leq \limsup_{j\to\infty}\left(\|x_{m_j} - x^\dagger\|^2 - \|x_{m_j+1} - x^\dagger\|^2 + \alpha_{k_j}C_2\right) \\
&= -\liminf_{j\to\infty}\left(\|x_{m_j+1} - x^\dagger\|^2 - \|x_{m_j} - x^\dagger\|^2\right) \\
&\leq 0.
\end{aligned}
$$

This implies that

$$
\lim_{j\to\infty}\|w_{m_j} - S_{m_j}w_{m_j}\| = 0 \qquad (23)
$$

and

$$
\lim_{j\to\infty}\|z_{m_j} - S_{m_j}z_{m_j}\| = 0. \qquad (24)
$$

Again, from (5), we have

$$
\begin{aligned}
\|w_{m_j} - x_{m_j}\| &= \|x_{m_j} + \psi_{m_j}(x_{m_j} - x_{m_j-1}) - \alpha_m[x_{m_j} + \psi_m(x_{m_j} - x_{m_j-1})] - x_{m_j}\| \\
&\leq \psi_{m_j}\|x_{m_j} - x_{m_j-1}\| + \alpha_{m_j}\|x_{m_j}\| + \alpha_m\psi_{m_j}\|x_{m_j} - x_{m_j-1}\| \\
&= \alpha_{m_j}\frac{\psi_{m_j}}{\alpha_{m_j}}\|x_{m_j} - x_{m_j-1}\| + \alpha_{m_j}\|x_{m_j}\| + \alpha_{m_j}^2\frac{\psi_{m_j}}{\alpha_{m_j}}\|x_{m_j} - x_{m_j-1}\| \to 0 \text{ as } j\to\infty. \qquad (25)
\end{aligned}
$$

Now, from (7) and (24), we have

$$\|x_{m_j+1} - z_{m_j}\| \leq (1 - \gamma_{m_j})\|z_{m_j} - S_{m_j} z_{m_j}\| \to 0 \text{ as } j \to \infty. \tag{26}$$

Using (26) and (24), we have

$$\|x_{m_j+1} - x_{m_j}\| \leq \|x_{m_j+1} - z_{m_j}\| + \|z_{m_j} - S_{m_j} z_{m_j}\| \to 0 \text{ as } j \to \infty. \tag{27}$$

Next, from (25) and (27), we have

$$\|w_{m_j} - x_{m_j+1}\| \leq \|w_{m_j} - x_{m_j}\| + \|x_{m_j+1} - x_{m_j}\| \to 0 \text{ as } j \to \infty. \tag{28}$$

By (26) and (28), we have

$$\|z_{m_j} - w_{m_j}\| \leq \|z_{m_j} - x_{m_j+1}\| + \|x_{m_j+1} - w_{m_j}\| \to 0 \text{ as } j \to \infty. \tag{29}$$

Since $\{x_m\}$ is bounded, a subsequence $\{x_{m_j}\}$ of $\{x_m\}$ exists such that $x_{m_j} \rightharpoonup \bar{u}$ as $j \to \infty$. By (25), we know that $w_{m_j} \rightharpoonup \bar{u}$ as $j \to \infty$. From (29), it follows that $z_{m_j} \rightharpoonup \bar{u}$ as $j \to \infty$. Since S_m satisfies the condition Z, then it implies from (24) that $\bar{u} \in \Omega$. Furthermore, since the sequence $\{x_{m_j}\}$ is bounded, then a subsequence $\{x_{m_{j_i}}\}$ of $\{x_{m_j}\}$ exists such that $x_{m_{j_i}} \rightharpoonup \bar{z} \in H$ as $i \to \infty$ and

$$\lim_{i\to\infty} \langle -x^\dagger, u_{m_{j_i}} - x^\dagger \rangle = \limsup_{j\to\infty} \langle -x^\dagger, x_{m_j} - x^\dagger \rangle. \tag{30}$$

Since $x^\dagger = P_\Omega(0)$, we get

$$\limsup_{j\to\infty} \langle -x^\dagger, x_{m_j} - x^\dagger \rangle = \lim_{i\to\infty} \langle -x^\dagger, x_{m_{j_i}} - x^\dagger \rangle = \langle -x^\dagger, \bar{z} - x^\dagger \rangle \leq 0. \tag{31}$$

From (27) and (31), we have

$$\limsup_{j\to\infty} \langle -x^\dagger, x_{m_j+1} - x^\dagger \rangle = \limsup_{j\to\infty} \langle -x^\dagger, x_{m_j} - x^\dagger \rangle = \langle -x^\dagger, \bar{z} - x^\dagger \rangle \leq 0. \tag{32}$$

Applying Lemma 2.4 to (20) and recalling (28), (32), (8) and the fact that $\lim_{j\to\infty} \alpha_{m_j} = 0$, we have that

$$\lim_{k\to\infty} \|x_m - u^\dagger\| = 0.$$

It follows that $\{x_m\}$ strongly converges to the point $x^\dagger = P_\Omega(0)$. □

5. Application to Convex Bilevel Optimization Problem

Bilevel optimization has received a lot of attention in recent years due to its applications in machine learning such as signal processing [23], image restoration [24], hyperparameter optimization [21,22] and reinforcement learning [25]. It is defined as a mathematical program in which an optimization problem contains another optimization problem as a constraint. In this part of the article, we consider the bilvel optimization problem in which the following minima are sought:

$$\min_{x\in T_*} \omega(x), \tag{33}$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ is strongly convex and differentiable, T_* is a nonempty set of inner level optimizers satisfying

$$\min_{x\in\mathbb{R}^m} \{\phi_1(x) + \phi_2(x)\}, \tag{34}$$

where $\phi_1 : \mathbb{R}^m \to \mathbb{R}$ is a differential and convex function such that $\nabla\phi_1(x)$ is L-Lipschitz continuous and $\phi_2 : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex, proper, and lower semi-continuous function. Let Γ denote the solution set of (33).

Notice that the bilevel optimization model contains the inner level minimization problem (34) as a constraint to the outer level optimization problem (33). It well known from (33) that

$$u^* \in \Gamma \text{ if and only if } \langle \nabla\omega(u^*), x - u^*\rangle\, \forall x \in T_*. \tag{35}$$

Several algorithms have been developed for solving the problem (34) [6,26]. The main algorithm is the proximal forward-backward approach, or proximal gradient method, defined by the iterative equation

$$x_{m+1} = pro_{\delta_m\phi_2}(I - \delta_m\nabla\phi_1)(x_m),\; m \in \mathbb{N}, \tag{36}$$

where $\delta_m > 0$ is the step-size, pro_{ϕ_2} is the proximity operator of ϕ_2, and $\nabla\phi_1$ is the gradient ϕ_1 [6]. Equation (36) is called the forward-backward splitting method algorithm (FBSA). This algorithm can be used to solve the inner level optimization problem if ϕ_1 is Lipschitz continuous [6].

On the other hand, the proximal gradient method can be seen as fixed point algorithm, where the iterated operator is given by

$$S := pro_{\delta_m\phi_2}(I - \delta\nabla\phi_1) \tag{37}$$

and is called the forward-backward mapping [27]. The forward-backward mapping, S, is well known to be nonexpansive if $0 < \delta < \frac{2}{L}$, where L is a Lipschitz constant of $\delta\nabla\phi_1$ and, in this case, $F(S) = argmin\{\phi_1(x) + \phi_2(x)\}$.

Next, we give our main results in this section as follows:

Theorem 5.1. *Let $\phi_1 : H \to \mathbb{H}$ be a convex and differentiable function such that $\nabla\phi_1$ is Lipschitz continuous with constant $L_{\phi_1} > 0$ and $\phi_2 : H \to (-\infty, \infty]$ are proper lower semi-continuous and convex functions. Let $\{c_m\} \subset (0, \frac{2}{L_{\phi_1}})$ with $c_m \to c$ as $m \to \infty$, where $c \in (0, \frac{2}{L_{\phi_1}})$. Suppose $\{x_m\}$ is the sequence generated by the following algorithm:*

Algorithm 5.2. Relax inertial Ishikawa iterative algorithm.
Initialization: Choose $\phi > 0$, $u_0, u_1 \in \mathcal{H}_1$, and set $m = 1$.

Iterative Steps: Calculate the next iteration point x_{m+1} as follows:

Step 1: Choose ψ_m such that $\psi_m \in [0, \bar{\psi}_m]$, where

$$\bar{\psi}_m = \begin{cases} \min\left\{\psi, \frac{\varepsilon_m}{\|x_m - x_{m-1}\|}\right\}, & \text{if } x_m \neq x_{m-1}, \\ \psi, & \text{otherwise.} \end{cases}$$

Step 2: Set

$$w_m = (1-\alpha)(a_m + \psi_m(a_m - a_{m-1}))$$

and compute

$$z_m = (1-\beta_m)w_m + \beta_m pro_{c_m\phi_2}(I - c_m\nabla\phi_1)w_m.$$

Step 3: Compute

$$x_{m+1} = (1-\gamma_m)z_m + \gamma_m pro_{c_m\phi_2}(I - c_m\nabla\phi_1)z_m. \tag{38}$$

Put $m := m+1$ and return to **Step 1**.

Suppose conditions (A_3)–(A_5) *holds, then* $\{x_m\}$ *converges strongly to an element solution of* (33).

Proof. Put $S_m = (I - c_m\nabla\phi_1)$, where $c_m = (0, \frac{2}{L_{\phi_1}})$. We also know that S_m is nonexpansive. Hence, the proof follows from that of Theorem 4.1. □

6. Numerical Experiment

In this section, we present an example to show the computational advantage of our suggested method by comparing it with some well known existing methods.

Example 6.1. Let $\mathbb{H} = C = \mathbb{R}$ and the mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ be defined by $Sx = \frac{x}{8}$. Let $\alpha_m = \beta_m = \theta_m = \gamma_m = \frac{1}{m+1}$, $\psi = 0.86$, $\varepsilon_m = \frac{100}{(m+1)^2}$ and choose the stopping criteria $E_m = \|x_{m+1} - x_m\| < 10^{-4}$. The obtained numerical results are demonstrated in the following table and figure:

Table 1. Numerical results of Example 6.1

	Algorithm 3.2	Picard-Ishikawa 2	Noor	Ishikawa	Mann
CPU time (sec.)	0.0020	0.0032	0.0035	0.0042	0.0042
No of Iter.	3	4	5	6	13

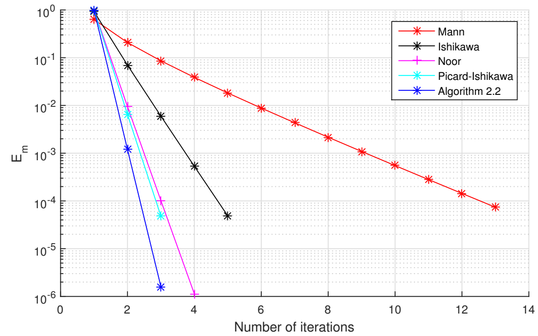

FIGURE 1. Graph of Table 1.

7. Conclusion

In this article, we have introduced an Ishikawa-type algorithm that employed inertial technique to enhance its speed of convergence. Under some standard assumptions, we proved the strong convergence results of the proposed method to the common solution of countable family of nonexpansive mapping. We applied the obtained results to solve convex optimization problems. We demonstrated the computational advantage of our method over some existing methods

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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