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ON AI-ITERATION PROCESS FOR FINDING FIXED POINTS OF ENRICHED CONTRACTION AND ENRICHED NONEXPANSIVE MAPPINGS WITH APPLICATION TO FRACTIONAL BVPs

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Abstract. In this article, we consider the AI-iteration process for approximating the fixed points of enriched contraction and enriched nonexpansive mappings. Firstly, we prove the strong convergence of the AI-iteration process to the fixed points of enriched contraction mappings. Furthermore, we present a numerical experiment to demonstrate the efficiency of the AI-iterative method over some existing methods. Secondly, we establish the weak and strong convergence results of AI-iteration method for enriched nonexpansive mappings in uniformly convex Banach spaces. Thirdly, the stability analysis results of the considered method is presented. Finally, we apply our results to the solution of fractional boundary value problems in Banach spaces.

Keywords: enriched contraction mapping; enriched nonexpansive mapping; stability; fractional BVPs.

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1. Introduction

The Banach contraction principle [1] is an essential tool for solving fixed points problems for contraction mappings defined on a complete metric space. This principle has widely been

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used by many authors for proving the existence and uniqueness of solutions of nonlinear functional equations such as integral equations, ordinary differential equations and partial differential equations. In certain cases, the existence of solutions of a fixed point problem is guaranteed, but finding the exact solution may be impossible [2]. The iterative method have been introduced by several authors to obtain the approximate solutions of such problems (see, for example, [9, 10, 11, 12, 13] and the references in them). It is important to mention that the proof of the Banach contraction principle is based on convergence of the most simplest iterative process called the sequence of successive approximations or Picard iterative process and it is well known that the approximation of this method to the fixed point of nonexpansive mappings may fail even when the fixed points of such mappings exist. Due to limitation of the Picard method, several methods have been constructed for approximating the fixed points of nonexpansive mappings and other generalizations of nonexpanisve mappings (see, [3, 2, 7, 4, 5, 14, 6, 33, 34, 35, 36, 37, 8] and references in them).

Let C be a nonempty subset of a Banach space B. Then the set $\{p \in C : p = Tp\}$ of all fixed points of the self mapping T defined C is denoted by $F(T)$. The mapping T is called a contraction if there exists $\delta \in [0,\infty)$ such that $\|Tx-Ty\| \leq \delta\|x-y\|$, for all $x,y \in C$. T is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|, \forall x,y \in C$ and it called quasinonexpansive if $F(T) \neq \emptyset$, then $\|Tx-p\| \leq \|x-p\|$, $\forall p \in F(T)$ and $x \in C$.

In [18], Berinde and Păcurar introduced a new class of mappings called enriched (b,γ)-contraction mappings.

Definition 1.1. *A mapping $T : C \to C$ is said to be an enriched (b,γ)-contraction mapping if for $x,y \in C$, there exists $b \in [0,\infty)$ and $\gamma \in (0,b+1)$ such that*

$$\|b(x-y)+Tx-Ty\| \leq \gamma\|x-y\|. \tag{1}$$

Remark 1.2. *It is not hard to see that, if $b = 0$, then the class of enriched (b,γ)-contraction mappings properly includes the class of contraction mappings.*

Again, Berinde [19] introduced another class of mappings called enriched nonexansive mappings.

Definition 1.3. *A mapping $T : C \to C$ is said to be an enriched nonexpansive mapping if for $x, y \in C$, there exists $b \in [0,\infty)$ such that*

$$\|b(x-y)+Tx-Ty\| \leq \|x-y\|. \tag{2}$$

Remark 1.4. *Observe that the class of enriched nonexpansive mappings is a super class of the class of nonexpansive mappings. Indeed, if $b=0$, it is obvious that the class of e properly includes the class of nonexpansive mappings.*

The class of enriched nonexpansive mappings have been studied by many researchers in recent years [19, 2, 20, 14].

Remark 1.5. [22] *Suppose T is a self mapping defined on C. Then, for any $\lambda \in [0,1)$, the averaged mapping T_λ on C given by*

$$T_\lambda x = (1-\lambda)x + \lambda Tx$$

satisfies $F(T) = F(T_\lambda)$. Obviously, $T_0 = I$ and $T_1 = T$ are the trivial cases.

Recently, Ofem and Igbokwe [23] introduced the AI-iterative method to approximate the fixed points of contraction mappings. The authors showed that their method converges faster than some well known methods in the literature. For $\{\sigma_n\} \in (0,1)$, the AI-iterative method is given as follows:

$$\begin{cases} p_1 \in C, \\ z_k = (1-\sigma_k)p_k + \sigma_k T_\lambda p_k, \\ w_k = T_\lambda z_k, & k \in \mathbb{N}. \\ q_k = T_\lambda w_k, \\ p_{k+1} = T_\lambda q_k, \end{cases} \tag{3}$$

Motivated by the above results, in this work, firstly, we prove the strong convergence of the AI-iteration process to the fixed points of enriched (b,γ)-contraction mappings. Furthermore, we present a numerical experiment to demonstrate the efficiency of AI-iterative method over some existing methods. Secondly, we establish the weak and strong convergence results of AI-iteration method for enriched nonexpansive mappings in uniformly convex Banach spaces.

Thirdly, we show the stability analysis results of the considered method. Finally, we apply our results to the solution of fractional boundary value problems in Banach spaces.

2. PRELIMINARIES

In this section, we recall some definitions and lemmas that will be used in obtaining our main results.

Definition 2.1. *Let B be a Banach space. If for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for $p,q \in B$ with $\|p\| \leq 1$, $\|q\| \leq 1$ and $\|p-q\| > \varepsilon$, implies $\left\|\frac{p+q}{2}\right\| < 1-\delta$. Then B is said to be a uniformly convex Banach space (UCBS).*

Definition 2.2. *A Banach B is said to fulfill the Opial's property if for any sequence $\{p_k\} \in B$ which converges weakly to $p \in B$ implies*

$$\limsup_{k\to\infty} \|p_k - p\| < \limsup_{k\to\infty} \|p_k - q\|, \forall q \in B \text{ with } q \neq p.$$

Definition 2.3. *Let $\{p_k\}$ be a bounded sequence in a Banach space B and let C be a nonempty closed convex subset of B. For $p \in B$, we take*

$$r(p,\{p_k\}) = \limsup_{k\to\infty} \|p_k - p\|.$$

The asymptotic radius of $\{p_k\}$ relative to C is given as:

$$r(C,\{p_k\}) = \inf\{r(p,\{p_k\}) : g \in C\}.$$

The asymptotic center of $\{p_k\}$ relative to C is defined by

$$A(C,\{p_k\}) = \{p \in C : r(p,\{p_k\}) = r(C,\{p_k\})\}.$$

It is well known that in a UCBS, the set $A(C,\{p_k\})$ is a singleton.

Definition 2.4. *Let B be a Banach space and C be a nonempty closed convex subset of B. Then, the self mapping $T : C \to C$ is said to demiclosed with respect to $p \in B$, if for each sequence $\{p_k\}$ which is weakly convergent to $p \in C$ and $\{Tp_k\}$ converges strongly to q implies that $Tp = q$.*

Lemma 2.5. [29] *Let B be a UCBS and $\{r_k\}$ be any sequence fulfilling $0 < h \leq r_k \leq q < 1$ for all $k \geq 1$. Suppose $\{p_k\}$ and $\{q_k\}$ are any sequences in B with*

$$\begin{aligned} \limsup_{k\to\infty} \|p_k\| &\leq z, \\ \limsup_{k\to\infty} \|q_k\| &\leq z \text{ and} \\ \limsup_{k\to\infty} \|r_k p_k + (1-r_k)q_k\| &= z \end{aligned}$$

hold for some $z \geq 0$. Then $\lim\limits_{k\to\infty} \|p_k - q_k\| = 0$.

Lemma 2.6. [28] *Let $\{a_k\}$ and $\{\omega_k\}$ be sequences of positive real numbers satisfying the following inequality:*

$$a_{k+1} \leq (1-\omega_k)a_k + e_k,$$

where $\omega_k \in (0,1)$ for all $k \in 0$ with $\sum_{k=0}^{\infty} \omega_k = \infty$. If $\lim\limits_{k\to\infty} \frac{e_k}{\omega_k}$, then $\lim\limits_{k\to\infty} a_k = 0$.

Definition 2.7. [31] *The condition (I) is said to be satisfied by the mapping $T : C \to C$, if a nondecreasing function $h : [0,\infty) \to [0,\infty)$ exists with $h(0) = 0$ and for all $c > 0$ then $h(c) > 0$ with $\|p - Tp\| \geq h(d(p, F(T))$ for all $p \in C$, where $d(p, F(T)) = \inf_{p^* \in F(T)} \|p - p^*\|$.*

Definition 2.8. [15] *Let t_k be an approximate sequence of p_k in a subset C of a Banach space B. Then a given iterative process $p_{k+1} = f(T, p_k)$ for some function f, converging to a fixed point p^* of self mapping T defined on C, is said to be T-stable or stable with respect to T provided that $\lim\limits_{k\to\infty} v_k = 0$ if and only if $\lim\limits_{k\to\infty} t_k = p^*$ where v_k is given by*

$$v_k = \|t_{k+1} - f(T, t_k)\|, \forall k \geq 1.$$

3. Convergence Analysis for (b,γ)-Contraction Mappings

In this section, we establish the convergence analysis of (b,γ)-contraction and enriched nonexpansive mappings.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a Banach space B and $T : C \to C$ a (b,γ)-contraction mapping with F(T)$\neq 0$. Then, the sequence $\{p_k\}$ defined by* (3) *converges to a fixed point of T.*

Proof. Let $b = \frac{1}{\lambda} - 1$, it follows that $\lambda \in (0,1)$. Then (1) becomes

$$\|(\frac{1}{\lambda} - 1)(p-q) + Tp - Tq\| \leq \gamma\|p-q\|, \tag{4}$$

which can equivalently be written as

$$\|T_\lambda p - T_\lambda q\| \leq \delta\|p-q\|, \tag{5}$$

where $\theta = \lambda\gamma$. As $\gamma \in (0, b+1)$, $\theta \in (0,1)$. Hence, the averaged operator T_λ is a contraction with contractive constant θ. Let $p^* \in F(T)$. Then, from (3) we have

$$\begin{aligned}
\|z_k - p^*\| &= \|(1-\sigma_k)p_k + \sigma_k T_\lambda p_k - p^*\| \\
&\leq (1-\sigma_k)\|p_k - p^*\| + \sigma_k\|T_\lambda p_k - p^*\| \\
&\leq (1-\sigma_k)\|p_k - p^*\| + \sigma_k\theta\|p_k - p^*\| \\
&= (1-(1-\theta)\sigma_k)\|p_k - p^*\|. \qquad (6)
\end{aligned}$$

Again, from (3), we have

$$\begin{aligned}
\|w_k - p^*\| &= \|T_\lambda z_k - p^*\| \\
&\leq \theta\|z_k - p^*\|. \qquad (7)
\end{aligned}$$

Also, from (3), we have

$$\begin{aligned}
\|q_k - p^*\| &= \|T_\lambda w_k - p^*\| \\
&\leq \theta\|w_k - p^*\|. \qquad (8)
\end{aligned}$$

Finally, from (3), we get

$$\begin{aligned}
\|p_{k+1} - p^*\| &= \|T_\lambda q_k - p^*\| \\
&\leq \theta\|q_k - p^*\|. \qquad (9)
\end{aligned}$$

Combining (6), (7),(8) and (9), we have

$$\|p_{k+1} - p^*\| \leq \theta^3(1-(1-\theta)\sigma_k)\|p_k - p^*\|. \tag{10}$$

Inductively, we have

$$\|p_{k+1}-p^*\| \leq \theta^{3k}(1-(1-\theta)\sigma_k)\|p_0-p^*\|. \tag{11}$$

Since $0<\theta^{3n}(1-(1-\theta)\sigma_n)<1$, it follows that $\{p_k\}$ converges to p^*. □

Next, we give an example to demonstrate that the AI-iteration process (3) converges faster than S [6], Thakur [30], and M [14] iteration methods.

Example 3.2. *Let $B=\mathbb{R}$ and $C=[0,10]$. Let $T:C\to C$ be a mapping defined by $Tp=10-p$, for all $p\in C$. Let all the control parameters in the compared methods be $\sigma_k=\frac{3}{4}$ and the starting point $p_1=6$. Observe that T is $\left(\frac{3}{5},\frac{5}{8}\right)$-enriched contraction with fixed point 5. Hence, $T_{\frac{5}{8}}=\frac{25-p}{4}$*

From Table 1 and Figure 1, it is evident that AI-iteration process converges faster to 5 than the compared methods.

TABLE 1. Convergence behaviour of various iterative methods for Example 3.2.

Step	S-iteration	Thakur	M-iteration	AI-iteration
1	6.0000000000	6.0000000000	6.0000000000	6.0000000000
2	5.4843750000	5.4316406250	5.2792968750	5.2421875000
3	5.2346191406	5.1863136292	5.0780067444	5.0586547852
4	5.1136436462	5.0804205313	5.0217870399	5.0142054558
5	5.0550461411	5.0347127684	5.0060850522	5.0034403838
6	5.0266629746	5.0149834411	5.0016995361	5.0008332180
7	5.0129148783	5.0064674619	5.0004746751	5.0002017950
8	5.0062556442	5.0027916193	5.0001325753	5.0000488722
9	5.0030300777	5.0012049763	5.0000370279	5.0000118362
10	5.0014676939	5.0005201167	5.0000103418	5.0000028666
11	5.0007109142	5.0002245035	5.0000028884	5.0000006943
12	5.0003443491	5.0000969048	5.0000008067	5.0000001681
13	5.0001667941	5.0000418281	5.0000002253	5.0000000007
14	5.0000807909	3.0000180547	5.0000000629	5.0000000000
15	5.0000391331	5.0000077931	5.0000000176	5.0000000000
16	5.0000189551	5.0000033638	5.0000000049	5.0000000000
17	5.0000091814	5.0000014520	5.0000000014	5.0000000000
18	5.0000044472	5.0000006267	5.0000000004	5.0000000000
19	5.0000021541	5.0000002705	5.0000000001	5.0000000000
20	5.0000010434	5.0000000008	5.0000000000	5.0000000000
21	5.0000005054	5.0000000000	5.0000000000	5.0000000000
22	5.0000002448	5.0000000008	5.0000000000	5.0000000000
23	5.0000001186	5.0000000000	5.0000000000	5.0000000000

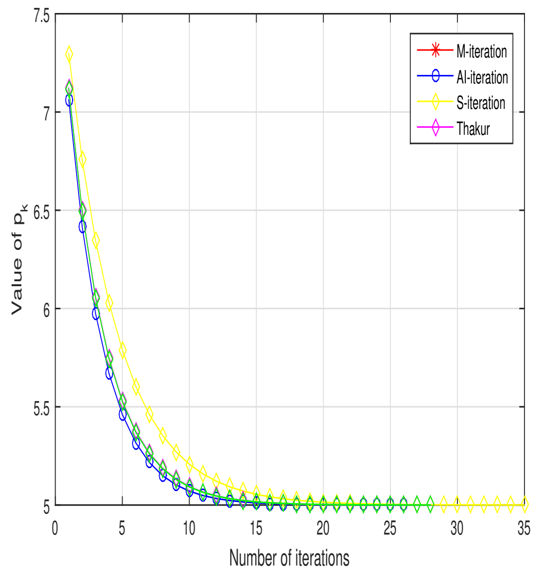

FIGURE 1. Graph corresponding to Table 1.

4. Convergence Analysis for Enriched Nonexpansive Mapping

In this section, we obtain the weak and strong convergence of AI-iteration method in uniformly convex Banach spaces.

Lemma 4.1. *Let C be a nonempty bounded closed convex subset of a UCBS B and $T : C \to C$ an enriched nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose p_k is a sequence generated by* (3)*, then $\lim_{k\to\infty} \|p_k - p^*\|$ exists for all $p^* \in F(T_\lambda)$.*

Proof. Since T is an enriched nonexpansive mapping, take $b = \frac{1}{\lambda} - 1$. It implies that $\lambda \in (0,1)$. From by (2), we have

$$\|(\frac{1}{\lambda} - 1)(p-q) + Tp - Tq\| \leq \|p-q\|, \tag{12}$$

which can be written in equivalent form as

$$\|T_\lambda p - T_\lambda q\| \leq \|p-q\|. \tag{13}$$

It means that the averaged operator T_λ is nonexpansive. Following the Browder's fixed point theorem, it follows that T_λ has at least one fixed point. By Remark 1.5, $F(T_\lambda) = F(T) = \emptyset$. We have shown that the averaged operator T_λ is nonexpansive. Now, let $p^* \in F(T_\lambda)$. From (3), we have

$$\begin{aligned}\|z_k - p^*\| &= \|(1-\sigma_k)p_k + \sigma_k T_\lambda p_k - p^*\| \\ &\leq (1-\sigma_k)\|p_k - p^*\| + \sigma_k\|T_\lambda p_k - p^*\| \\ &\leq (1-\sigma_k)\|p_k - p^*\| + \sigma_k\|p_k - p^*\| \\ &= \|p_k - p^*\|. \end{aligned} \tag{14}$$

Again, from (3), we have

$$\begin{aligned}\|w_k - p^*\| &= \|T_\lambda z_k - p^*\| \\ &\leq \|z_k - p^*\|. \end{aligned} \tag{15}$$

Also, from (3), we have

$$\begin{aligned}\|q_k - p^*\| &= \|T_\lambda w_k - p^*\| \\ &\leq \|w_k - p^*\|. \end{aligned} \tag{16}$$

Finally, from (3), we get

$$\begin{aligned}\|p_{k+1} - p^*\| &= \|T_\lambda q_k - p^*\| \\ &\leq \|q_k - p^*\|. \end{aligned} \tag{17}$$

Combining (14), (15),(16) and (17), we have

$$\|p_{k+1} - p^*\| \leq \|p_k - p^*\|. \tag{18}$$

There, $\{\|p_k - p^*\|\}$ is a bounded monotone decreasing sequence. It follows that, $\lim_{k\to\infty}\|p_k - p^*\|$ exists for all $p^* \in F(T) = F(T_\lambda)$. $\square$

Theorem 4.2. *Let B, C, T and $\{p_k\}$ be same as in Lemma 4.1. Then, $F(T) \neq \emptyset$ if and only if $\{p_k\}$ is bounded and $\lim_{k\to\infty}\|p_k - T_\lambda p_k\| = 0$, where $b = \frac{1}{\lambda} - 1$.*

Proof. By Lemma 4.1, it is shown that $\{p_k\}$ is bounded and $\lim_{k\to\infty}\|p_k - p^*\|$ exists for all $p^* \in F(T_\lambda)$. Suppose

$$\lim_{k\to\infty}\|p_k - p^*\| = h. \tag{19}$$

From (19) and (14), we have

$$\limsup_{k\to\infty}\|z_k - p^*\| \leq \limsup_{k\to\infty}\|p_k - p^*\| = h. \tag{20}$$

By (19), we obtain

$$\limsup_{k\to\infty}\|T_\lambda p_k - p^*\| \leq \limsup_{k\to\infty}\|p_k - p^*\| = h. \tag{21}$$

Recalling (3), we have

$$\begin{aligned}
\|p_{k+1} - p^*\| &= \|T_\lambda q_k - p^*\| \\
&\leq \|q_k - p^*\| \\
&= \|T_\lambda w_k - p^*\| \\
&\leq \|w_k - p^*\| \\
&= \|T_\lambda z_k - p^*\| \\
&\leq \|z_k - p^*\|.
\end{aligned}$$

Therefore, from (19), we have

$$h \leq \liminf_{k\to\infty}\|z_k - p^*\|. \tag{22}$$

From (22) and (20), we have

$$\begin{aligned}
h &= \lim_{k\to\infty}\|z_k - p^*\| \\
&= \lim_{k\to\infty}\|(1-\sigma_k)p_k + \sigma_k T_\lambda p_k - p^*\| \\
&= \lim_{k\to\infty}\|(1-\sigma_k)(p_k - p^*) + \sigma_k(T_\lambda p_k - p^*)\| \\
&= \lim_{k\to\infty}((1-\sigma_k)\|p_k - p^*\| + \sigma_k\|T_\lambda p_k - p^*\|) \\
&\leq \lim_{k\to\infty}((1-\sigma_k)\|p_k - p^*\| + \sigma_k\|p_k - p^*\|) \\
&\leq h. \qquad (23)
\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \|(1-\sigma_k)(p_k-p^*)+\sigma_k(T_\lambda p_k - p^*)\| = h. \tag{24}$$

Using (19), (21), (24) and Lemma 2.5, we obtain

$$\lim_{k\to\infty} \|p_k - T_\lambda p_k\| = 0. \tag{25}$$

On the other hand, suppose $\{p_k\}$ is bounded and $\lim_{k\to\infty} \|p_k - Tp_k\| = 0$. Let $p^* \in A(C,\{p_k\})$. From 13, we get

$$\begin{aligned} r(T_\lambda p^*, \{p_k\}) &= \limsup_{k\to\infty} \|p_k - T_\lambda p^*\| \\ &\leq \limsup_{k\to\infty} \|p_k - T_\lambda p_k\| + \limsup_{k\to\infty} \|T_\lambda p_k - p^*\| \\ &= \limsup_{k\to\infty} \|p_k - p^*\| \\ &= r(p^*, \{p_k\}). \end{aligned}$$

It means that $T_\lambda p^* \in A(C,\{p_k\})$. Since B is a UCBS, it implies that $A(C,\{p_k\})$ is a singleton set and therefore, we have that $T_\lambda p^* = p^*$. Thus, $F(T_\lambda) \neq \emptyset$. □

Theorem 4.3. *Let B, C, T and $\{p_k\}$ be same as in Lemma 13 such that $F(T) \neq \emptyset$. Assume that B fulfills the Opial's property, then the AI iterative scheme $\{p_k\}$* (3) *weakly converges to an element in $F(T)$.*

Proof. For $F(T_\lambda) \neq \emptyset$, it is shown in Theorems 4.1 and 4.2 that $\lim_{k\to\infty} \|p_k - p^*\|$ exists and $\lim_{k\to\infty} \|p_k - T_\lambda p_k\| = 0$. In what follows, we will show the impossibility of $\{p_k\}$ to posses two weak sub-sequential limits in $F(T_\lambda)$. Let c and d be two weak sub-sequential limits of $\{p_{k_i}\}$ and $\{p_{k_j}\}$, respectively. Thanks to Theorem 4.2, we have that $(I - T_\lambda)$ is demiclosed at 0,it follows that $(I - T_\lambda)c = 0$. Therefore, $T_\lambda c = c$. Using similar approach, we can prove that $T_\lambda d = d$. Now, we prove uniqueness. Suppose $c \neq d$, then by Opial's condition

$$\begin{aligned} \lim_{k\to\infty} \|p_k - c\| &= \lim_{i\to\infty} \|p_{k_i} - c\| < \lim_{i\to\infty} \|p_{k_i} - d\| = \lim_{k\to\infty} \|p_k - d\| \\ &= \lim_{j\to\infty} \|p_{k_j} - d\| < \lim_{j\to\infty} \|p_{k_j} - c\| = \lim_{k\to\infty} \|p_k - c\|, \end{aligned}$$

which is a contradiction, therefor $c = d$. Thus, $\{p_k\}$ weakly converges to $c \in F(T_\lambda)$. □

Now, we present some strong convergence Theorems.

Theorem 4.4. *Let B, C, T_λ and $\{p_k\}$ be same as in Theorem 4.1 such that $F(T_\lambda) \neq \emptyset$. Then, $\{p_k\}$ converges strongly to an element in $F(T_\lambda)$ if and only if* $\liminf_{k\to\infty} d(p_k, F(T_\lambda)) = 0$, *where* $d(p_k, F(T_\lambda)) = \inf\{\|p_k - p^*\| : p^* \in F(T_\lambda)\}$.

Proof. The necessity case is trivial. Thus, we consider only the sufficient case. Assume that $\liminf_{k\to\infty} d(p_k, F(T_\lambda)) = 0$ and $p^* \in F(T_\lambda)$. Then, by Theorem 13, we have that $\lim_{k\to\infty} \|p_k - p^*\|$ exists, for each $p^* \in F(T_\lambda)$. It is now enough to prove that $\{p_k\}$ is a Cauchy sequence in C. Due to $\lim_{k\to\infty} d(p_k, F(T_\lambda)) = 0$, then for $\xi > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $k \geq m_0$

$$\begin{aligned} d(p_k, F(T_\lambda)) &< \frac{\xi}{2} \\ \inf\{\|p_k - p^*\| : p^* \in F(T_\lambda)\} &< \frac{\xi}{2}. \end{aligned}$$

In particular, $\inf\{\|p_{m_0} - p^*\| : p^* \in F(T_\lambda)\} < \frac{\xi}{2}$. Therefore, there exists $p^* \in F(T_\lambda)$ such that

$$\|p_{m_0} - p^*\| < \frac{\xi}{2}.$$

If $m, k \geq m_0$, we have

$$\begin{aligned} \|p_{k+l} - p_k\| &\leq \|p_{k+l} - p^*\| + \|p_k - p^*\| \\ &\leq \|p_{m_0} - p^*\| + \|p_{m_0} - p^*\| \\ &= 2\|p_{m_0} - p^*\| < \xi. \end{aligned}$$

This means that the sequence $\{p_k\}$ is Cauchy in C. Since C is closed, it follows that a point $t \in C$ with $\lim_{k\to\infty} p_k = t$. So that $\lim_{k\to\infty} d(p_k, F(T_\lambda)) = 0$ implies that $d(t, F(T_\lambda)) = 0$, that is, $t \in F(T_\lambda)$. □

Theorem 4.5. *Let B, C, T_λ and $\{p_k\}$ be same as in 4.1 such that $F(T_\lambda) \neq \emptyset$. Assume C is a nonempty convex compact subset of B, then $\{p_k\}$ strongly converges to an element in $F(T_\lambda)$.*

Proof. Thanks to Theorem 4.2, it is shown that $\lim_{k\to\infty} \|p_k - T_\lambda p_k\| = 0$. By the compactness of C, it follows that $\{p_k\}$ has a strong convergent subsequence $\{p_{k_i}\}$ with a strong limit c. Hence,

$$\begin{aligned}\|p_{k_i}-T_\lambda c\| &\leq \|p_{k_i}-T_\lambda p_{k_i}\|+\|T_\lambda p_{k_i}-T_\lambda c\|\\ &\leq \|p_{k_i}-T_\lambda p_{k_i}\|+\|p_{k_i}-c\|.\end{aligned}$$

Letting $i\to\infty$, we get $p_{k_i}\to T_\lambda c$. Thus, $T_\lambda c=c$, i.e. $c\in F(T_\lambda)$. From Theorem 4.1, we know that $\lim_{k\to\infty}\|p_k-c\|$ exists. In what follows, we have that c is a strong limit for $\{p_k\}$. □

Theorem 4.6. *Let B, C, T_λ and $\{p_k\}$ be same as in 4.1 such that $F(T_\lambda)\neq\emptyset$. If T_λ satisfies the condition I, then $\{p_k\}$ converges strongly to an element in $F(T_\lambda)$.*

Proof. It shown in Theorem 4.2 that

$$\lim_{k\to\infty}\|p_k-T_\lambda p_k\|=0. \tag{26}$$

By (26) and Definition 2.7, we have

$$0\leq\lim_{k\to\infty}f(d(p_k,F(T)))\leq\lim_{k\to\infty}\|p_k-Tp_k\|=0\ \Rightarrow\ f(d(p_k,F(T)))=0.$$

Since the function $h:[0,\infty)\to[0,\infty)$ nondecreasing such that $h(0)=0$ and $h(g)>0$, for all $g>0$, we have

$$\lim_{k\to\infty}d(p_k,F(T))=0.$$

Using Theorem 4.4, the remainder of the proof is obtained. □

5. Stability Result

In this section, we present the stability result of the AI iterative method 3.

Theorem 5.1. *Let C be a nonempty closed and convex subset of a Banach space B and $T:C\to C$ a (b,γ)-enriched contraction mapping. Then, the AI iterative scheme defined by* (3) *is T_λ-stable for $\lambda=\frac{1}{\lambda}-1$.*

Proof. Let t_k be an approximate sequence of $\{p_k\}$ in C. The sequence defined by iteration (3) is give by $p_{k+1}=f(T_\lambda,p_k)$ and $v_k=\|t_{k+1}-f(T_\lambda,t_k)\|$, for all $k\geq1$. Next, we have to show that $\lim_{k\to\infty}v_k=0$ if and only if $\lim_{k\to\infty}a_k=p^*$. From (3), we have that

$$\begin{aligned}\|t_{k+1}-p^*\| &\leq \|t_{k+1}-f(T_\lambda,t_k)\|+\|f(T_\lambda,t_k)-p^*\|\\ &= v_k+\|p_{k+1}-p^*\|.\end{aligned}$$

By (4.1), we have

$$\|t_{k+1}-p^*\| \leq v_k+\theta^3(1-(1-\theta)\sigma_k)\|t_k-p^*\|.$$

Take $a_k=\|t_k-p^*\|$ and $\omega_k=\theta^3(1-(1-\theta)\sigma_k)$. Then we have $a_{k+1}=\theta^3(1-\omega_k)a_k+v_k$. Since $\lim\limits_{k\to\infty}\frac{v_k}{\omega_k}=0$, it follows from Lemma 2.6 that we have $\lim\limits_{k\to\infty}t_k=0$ and hence, $\lim\limits_{k\to\infty}t_k=p^*$

Conversely, if $\lim\limits_{k\to\infty}t_k=p^*$, then we have

$$\begin{aligned}v_k &= \|t_{k+1}-f(T_\lambda,t_k)\|\\ &\leq \|t_{k+1}-p^*\|+\|f(T_\lambda,t_k)-p^*\|\\ &\leq \|t_{k+1}-p^*\|+\theta^3(1-(1-\theta)\sigma_k)\|t_k-p^*\|.\end{aligned}$$

This implies that $\lim\limits_{k\to\infty}v_k=0$. Hence, the AI iterative scheme (3) is T_λ-stable. □

6. APPLICATION

Fractional calculus is thought-about as a generalization of classical calculus. There are numerous definitions for derivatives and integrals of arbitrary order. Even though in the beginning, fractional calculus was just a strictly mathematical idea, in modern times its use has unfolded into many distinct fields of technological know-how such as mechanics, physics, biology, chemistry, engineering, electrochemistry and bioengineering [25].

The fractional differential equations have emerged as a new branch of applied mathematics, due to the evolution of fractional calculus. Further, the existence and uniqueness of solutions to fractional boundary value problems have acquired a lot of interest due to its qualitative properties of fractional differential equations [26, 27].

In this paper, we consider the following fractional boundary value problem with ψ-Caputo fractional derivative:

(27)
$$\begin{cases} {}^{c}D_{a^+}^{\alpha,\psi}p(t) = g(t,p(t)), \ t \in [a,b], \\ p_a^{[m]} = p_a^m, \ m = 0,1,...,k-2; \ p_{\psi}^{[k-1]}(b) = p_b, \end{cases}$$

where ${}^{c}D_{a^+}^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order $k-1 < \alpha < k$ ($k = [\alpha]+1$), and $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is the given continuous function and $p_b, p_a^m \in \mathbb{R}$ ($m = 0,1,...,k-2$), $p \in C^{k-1}[a,b]$ such that ${}^{c}D_{a^+}^{\alpha,\psi}p$ exists and is continuous in $[a,b]$.

The following definitions and lemmas will be useful in obtaining our main results in this part of the paper:

Definition 6.1. *Let $\alpha > 0$, h an integral functional function defined on $[a,b]$ and $\psi \in C^k[a,b]$ an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left-sided ψ-Riemann-Liouville fractional of order α of a function h is given by*

$$I_{a^+}^{\alpha,\psi}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds,$$

where $\Gamma(\cdot)$ is gamma function.

Definition 6.2. *Let $k-1 < \alpha < k$, $h : [a,b] \to \mathbb{R}$ be an integrable function and ψ is defined as in Definition 6.1. The left-sided ψ-Riemann-Liouville fractional derivative of order α of a function h is given by*

(28)
$$D_{a^+}^{\alpha,\psi}h(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k {}^{c}I_{a^+}^{k-\alpha,\psi}h(t),$$

where $k = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 6.3. *Let $k-1 < \alpha < k$, $h \in C^{k-1}[a,b]$ and ψ be defined as in Definition 6.1. The left-sided ψ-Riemann-Liouville fractional derivative of function h of order α is evaluated as*

(29)
$${}^{c}D_{a^+}^{\alpha,\psi}h(t) = I_{a^+}^{\alpha,\psi}h(t)\left[h(t) - \sum_{m=0}^{m-1}\frac{h_{\psi}^{[m]}(a)}{m!}(\psi(t)-\psi(a)^m)\right],$$

where $h_{\psi}^{[m]}(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^m h(t)$ and $k = [\alpha]+1$ for $\alpha \notin \mathbb{N}$, $k = \alpha$ for $\alpha \in \mathbb{N}$. Further, if $h \in C^k[a,b]$ and $\alpha \notin \mathbb{N}$, then

$${}^{c}D_{a^+}^{\alpha,\psi}h(t) = I_{a^+}^{k-\alpha,\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k h(t) = \frac{1}{\Gamma(k-\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{k-\alpha-1}h_{\phi}^{[k]}(s)ds,$$

Therefore, if $\alpha = k \in \mathbb{N}$, one get

$$^{c}D^{\alpha,\psi}_{a^+}h(t) = h^{[k]}_{\phi}(t).$$

Lemma 6.4. *A function p is a solution of the fractional boundary value problem* (27) *if and only if $p(t)$ is a solution of the fractional integral equation*

$$\begin{aligned} p(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,p(s))ds + \Bigg[\frac{p_b}{(k-1)!} \\ &+ \frac{g(a,p(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\Bigg] - \frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)} \\ &+ \int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,p(s))ds + \sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m. \end{aligned}$$

The existence and uniqueness results of the problem (27) was proved in [25] under the following assumptions:

H_1 $g:[a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a $0<\mu<1$ such that

$$|g(t,z)-g(t,w)| \leq \mu|z-w|, \forall z,w \in \mathbb{R}$$

H_2 $\left[\frac{1}{(\alpha+1)} + \frac{((\psi(b)-\psi(a))+k-1)}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha} < 1.$

Our aim in this part of the paper is approximate the solution of the fractional boundary value problem (27) using the AI iterative method (3) for $\lambda = 1$.

Theorem 6.5. *Suppose assumption H_1 –H_2 holds. Then the AI-iterative method defined by* (3) *converges to the solution of the BVP* (27).

Proof. Set $B = \{p \in C^{k-1}[a,b] :^{c} D^{\alpha,\psi}_{a^+}p \in C[a,b]\}$. Then B is a Banach space. Next, we define the operator $T : B \to B$ by

$$\begin{aligned} (Tp)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,p(s))ds + \Bigg[\frac{p_b}{(k-1)!} \\ &+ \frac{g(a,p(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\Bigg] - \frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)} \\ &+ \int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,p(s))ds + \sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m. \end{aligned} \tag{30}$$

Let $\{p_k\}$ be an iterative method generated by the AI-iterative method (3) for the operator defined by (30). We have to show that $p_k \to p^*$ as $k \to \infty$. By (3) and (30), we have

$$
\begin{aligned}
& \|p_{k+1}-p^*\| \\
= \ & |(Tq_k)(t)-(Tp^*)(t)| \\
= \ & \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,q_k(s))ds+\left[\frac{p_b}{(k-1)!}+\frac{g(a,q_k(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\right] \\
& -\frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)}+\int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,q_k(s))ds+\sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m \\
& -\left[\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,p^*(s))ds+\left[\frac{p_b}{(k-1)!}+\frac{g(a,p^*(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\right]\right. \\
& \left.-\frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)}+\int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,p^*(s))ds+\sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m\right] \\
\leq \ & I_{a^+}^{\alpha,\psi}|g(t,q_k(t))-g(t,p^*(t))|+\frac{(\psi(b)-\psi(a))^{\alpha-k+1}(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}|g(a,q_k(a))-g(a,p^*(a))| \\
& +\frac{(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}I_{a^+}^{\alpha-k+1,\psi}|g(b,q_k(b))-g(b,p^*(b))| \\
\leq \ & I_{a^+}^{\alpha,\psi}\mu|p_k(t)-p^*(t)|+\frac{(\psi(b)-\psi(a))^{\alpha-k+1}(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}\mu|p_k(a)-p^*(a)| \\
& +\frac{(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}I_{a^+}^{\alpha-k+1,\psi}\mu|p_k(b)-p^*(b)| \\
\leq \ & \left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{(k-2)!\Gamma(\alpha-k+2)}+\frac{(\psi(b)-\psi(a))^{\alpha+1}}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu\|q_k-p^*\|.
\end{aligned}
$$

Thus, we have

$$\|p_{k+1}-p^*\| \leq \left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\|q_k-p^*\|. \tag{31}$$

Following similar approach, from (3) we obtain

$$\|q_k-p^*\| \leq \left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\|w_k-p^*\|. \tag{32}$$

$$\|w_k-p^*\| \leq \left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\|z_k-p^*\|. \tag{33}$$

Using (3), we have

$$
\begin{aligned}
& \|z_k(t)-p(t)\| \\
= \ & |(1-\sigma_k)p_k(t)+\sigma_k(Tq_k)(t)-p^*(t)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\sigma_k)|p_k(t)-p^*(t)|+\sigma_k|(Tq_k)(t)-(Tp^*)(t)| \\
&= (1-\sigma_k)|p_k(t)-p^*(t)|+\sigma_k\times \\
&\Bigg\{\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,p_k(s))ds+\left[\frac{p_b}{(k-1)!}+\frac{g(a,p_k(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\right] \\
&-\frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)}+\int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,p_k(s))ds+\sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m \\
&-\Bigg(\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,p^*(s))ds+\left[\frac{p_b}{(k-1)!}+\frac{g(a,p^*(a))(\psi(b)-\psi(a))^{\alpha-k+1}}{(k-2)!\Gamma(\alpha-k+2)}\right] \\
&-\frac{(\psi(t)-\psi(a))^{k-1}}{(k-1)!\Gamma(\alpha-k+1)}+\int_a^t \psi'(s)(\psi(b)-\psi(s))^{\alpha-k}g(s,p^*(s))ds+\sum_{m!}^{k-2}\frac{p_a^m}{m!}[\psi(t)-\psi(a)]^m\Bigg)\Bigg\} \\
&\leq (1-\sigma_k)|p_k(t)-p^*(t)|+\sigma_k\times \\
&\Bigg\{I_{a^+}^{\alpha,\psi}|g(t,p_k(t))-g(t,p^*(t))|+\frac{(\psi(b)-\psi(a))^{\alpha-k+1}(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}|g(a,p_k(a))-g(a,p^*(a))| \\
&+\frac{(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}I_{a^+}^{\alpha-k+1,\psi}|g(b,p_k(b))-g(b,p^*(b))|\Bigg\} \\
&\leq (1-\sigma_k)|p_k(t)-p^*(t)|+\sigma_k\times \\
&\Bigg\{I_{a^+}^{\alpha,\psi}\mu|p_k(t)-p^*(t)|+\frac{(\psi(b)-\psi(a))^{\alpha-k+1}(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}\mu|p_k(a)-p^*(a)| \\
&+\frac{(\psi(t)-\psi(a))^{k-1}}{(k-2)!\Gamma(\alpha-k+2)}I_{a^+}^{\alpha-k+1,\psi}\mu|p_k(b)-p^*(b)|\Bigg\} \\
&\leq (1-\sigma_k)\|p_k-p^*\| \\
&+\sigma_k\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{(k-2)!\Gamma(\alpha-k+2)}+\frac{(\psi(b)-\psi(a))^{\alpha+1}}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu\|p_k-p^*\|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|z_k-p^*\| &\leq (1-\sigma_k)\|p_k-p^*\| \\
&\quad+\sigma_k\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\|p_k-p^*\| \\
&= \left(1-\sigma_k\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\right)\|p_k-p^*\|. \qquad (34)
\end{aligned}
$$

Combing (31), (32), (33) and (34), we have

$$
\begin{aligned}
\|p_{k+1}-p^*\| &\leq \left(\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\right)^3\times \\
&\quad\left(1-\sigma_k\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\right)\|p_k-p^*\|. \qquad (35)
\end{aligned}
$$

Since $\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}<1$ and $0<\sigma_k<1$, it follows that $\left(1-\sigma_k\left[\frac{1}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))+k-1}{(k-1)!\Gamma(\alpha-k+2)}\right]\mu(\psi(b)-\psi(a))^{\alpha}\right)<1$. Thus, (35) becomes

$$\|p_{k+1}-p^*\|\leq\|p_k-p^*\|. \tag{36}$$

If we set, $\|p_{k+1}-p^*\|=v_k$, then we have

$$v_{k+1}\leq v_k,\forall k\geq 1. \tag{37}$$

Thus, $\{v_k\}$ is a monotone decreasing sequence of positive real numbers. Further, it is bounded sequence, we obtain

$$\lim_{k\to\infty} v_k=\inf\{v_k\}=0.$$

So,

$$\lim_{k\to\infty}\|p_k-p^*\|=0.$$

□

7. CONCLUSION

In this article, we considered the AI-iterative method for approximating the fixed pints of enriched (b,γ)-contraction mappings and enriched nonexpansive mappings. We obtained the weak and strong convergence results of these mappings under some mild conditions. we present a numerical example to justify the advantage of AI-iterative method over many existing methods. Furthermore, we showed that the AI-iterative method is T-stable. Lastly, we approximate the solution of fractional BVPs with via AI-iterative method.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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