AIP | Physics of Plasmas

Nonlinear, stationary electrostatic ion cyclotron waves: Exact solutions for solitons, periodic waves, and wedge shaped waveforms

J. F. McKenzie, T. B. Doyle, and S. S. Rajah

Citation: Phys. Plasmas **19**, 112115 (2012); doi: 10.1063/1.4769031
View online: http://dx.doi.org/10.1063/1.4769031
View Table of Contents: http://pop.aip.org/resource/1/PHPAEN/v19/i11
Published by the American Institute of Physics.

Nonlinear, stationary electrostatic ion cyclotron waves: Exact solutions for solitons, periodic waves, and wedge shaped waveforms

J. F. McKenzie,[1,2,a)] T. B. Doyle,[3,b)] and S. S. Rajah[1,c)]
[1]*Department of Mathematics, Statistics and Physics, Durban University of Technology, Steve Biko Campus, Durban 4001, South Africa*
[2]*School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag: X54001, Durban 4001, South Africa*
[3]*Materials Research Division, iThemba LABS, P.O.Box 722, Somerset West, 7129, South Africa and School of Chemistry and Physics, University of KwaZulu-Natal, Private Bag: X54001, Durban 4001, South Africa*

(Received 3 September 2012; accepted 8 November 2012; published online 30 November 2012)

The theory of fully nonlinear stationary electrostatic ion cyclotron waves is further developed. The existence of two fundamental constants of motion; namely, momentum flux density parallel to the background magnetic field and energy density, facilitates the reduction of the wave structure equation to a first order differential equation. For subsonic waves propagating sufficiently obliquely to the magnetic field, soliton solutions can be constructed. Importantly, analytic expressions for the amplitude of the soliton show that it increases with decreasing wave Mach number and with increasing obliquity to the magnetic field. In the subsonic, quasi-parallel case, periodic waves exist whose compressive and rarefactive amplitudes are asymmetric about the "initial" point. A critical "driver" field exists that gives rise to a soliton-like structure which corresponds to infinite wavelength. If the wave speed is supersonic, periodic waves may also be constructed. The aforementioned asymmetry in the waveform arises from the flow being driven towards the local sonic point in the compressive phase and away from it in the rarefactive phase. As the initial driver field approaches the critical value, the end point of the compressive phase becomes sonic and the waveform develops a wedge shape. This feature and the amplitudes of the compressive and rarefactive portions of the periodic waves are illustrated through new analytic expressions that follow from the equilibrium points of a wave structure equation which includes a driver field. These expressions are illustrated with figures that illuminate the nature of the solitons. The presently described wedge-shaped waveforms also occur in water waves, for similar "transonic" reasons, when a Coriolis force is included. © *2012 American Institute of Physics.* [http://dx.doi.org/10.1063/1.4769031]

I. INTRODUCTION

Electrostatic ion cyclotron waves have in part, because of their ubiquitous occurrence and manifold behaviours in both natural and laboratory magnetized plasmas, been the subject of numerous studies. A brief, but adequate, review of the field with some contextual relevance to the present work can be found in the paper by Reddy *et al.*[1] More recent publications of potential interest include those of Shin *et al.*[2] on observations of solitary waves by the Geotail space probe, Richardson *et al.*[3] and Burrows *et al.*[4] on solar wind and heliospheric termination shock detected by the Voyager probe, and Wilson *et al.*[5] on waves in interplanetary shocks. The objectives of the present analysis are to further develop and to clarify an earlier theoretical treatment (McKenzie[6]) on nonlinear electrostatic waves in a magnetized plasma, for which one of the motivations was to elucidate the mechanism giving rise to the "spiky" or cusp-like waveforms found from numerical solution of the equations (Reddy *et al.*[1]). In the present treatment, we show that periodic wedge-shaped (rather than cusp-shaped) waves, propagating at supersonic speeds, are, in fact, formed if the amplitude of the driver field equals a critical value corresponding to the minimum value of the structure function attained at the local sonic point of the flow. It is this transonic choked flow property, which arises in many wave systems (McKenzie *et al.*[7]), that provides the wave with its limiting wedge-shape. This phenomenon also arises in nonlinear water waves if a Coriolis force is present (see, for example, Shrira[8] and McKenzie[9]). Periodic waves, propagating quasi-parallel to the magnetic field, can also be constructed and also have a limiting form which arises when the energy of the driver field equals another/different minimum of the structure function.

The wave structure equation, in the weakly nonlinear limit (obtained by expanding the Bernoulli momentum and energy functions) yields the classical soliton sech^2 hump of compression for obliquely propagating "subsonic" waves. The fully nonlinear treatment given herein yields the exact solution for the wave amplitude for the soliton as a function of the Mach number for various obliquity angles. This nonlinear treatment generalises earlier weakly nonlinear theories and predicts the important result, as demonstrated by exact solutions for the equilibrium point corresponding to the centre of the wave, that the strength of the soliton increases with obliquity and with decreasing Mach number. This result is not obtainable from weakly nonlinear theories.

a)mckenziej@ukzn.ac.za.
b)doyle@tlabs.ac.za.
c)misthrys@dut.ac.za.

In Sec. II, the governing equations, with particular attention to the two fundamental constants of the motion (namely, momentum flux and energy density), which enable the wave system to be reduced to a first order differential equation for the longitudinal flow speed, will be recalled. Exact (implicit) expressions for the wave amplitudes, as a function of Mach number and obliquity, are given. In Sec. III, subsonic soliton solutions are discussed and in Sec. IV periodic solutions. The results are summarized in Sec. V.

II. THE WAVE STRUCTURE EQUATION

A fully nonlinear description of electrostatic ion cyclotron waves, propagating along the x-axis at an angle θ to the background magnetic field $B_0 = B_0(\cos\theta, \mathbf{0}, \sin\theta)$, has been developed by McKenzie,[6] extending the earlier weakly nonlinear treatments of Temerin *et al.*,[10] Yu *et al.*,[11] Jovanic and Shukla[12] and the later contributions by Reddy *et al.*,[1] McKenzie,[13] and McKenzie and Doyle.[14] In this description, it was shown that the existence of two important constants of motion, namely conservation of momentum parallel to B_0 and conservation of energy, facilitates the reduction of the wave structure problem to a first-order differential equation for the longitudinal component (parallel to the x-axis) of the wave flow velocity.

Here we briefly summarize the theory and reiterate the relevant equations.

The component of the cold ion equation parallel to B_0 immediately yields the integral expressing conservation of momentum parallel to B_0, namely,

$$P(u_{ix})\cos\theta + u_{iz}\sin\theta = const, \tag{1}$$

in which

$$P(u_{ix}) = u_{ix} + \frac{p_e}{M_i}. \tag{2}$$

Here, $P(u_{ix})$ is the momentum flux density in the x-direction (i.e., the dynamic pressure u_{ix} plus electron pressure p_e divided by the conserved ion mass flux density, $M_i = m_i n_{i0} U$). $\mathbf{u}_i = (u_{ix}, u_{iy}, u_{iz})$ is the flow velocity in the wave, n_{i0} is the unperturbed ion (electron) density, and U is the wave speed. The x-gradient of the electron pressure p_e is equal to the volume electric field force, $-en_eE_x$. Since the electrons are treated as massless there also exists a constant motional electric field $E_y\ (= UB_{0z})$ which implies that

$$(u_{ix} - U)B_{0z} = u_{iz}B_{0x}. \tag{3}$$

The second conserved quantity, namely energy, is obtained by taking the scalar product of the ion equation of motion with the ion velocity $\mathbf{u_i}$, yielding

$$\epsilon(u_{ix}) - UP(u_{ix}) + \frac{1}{2}(u_{iy}^2 + u_{iz}^2) = const, \tag{4}$$

where the "longitudinal" energy density is given by

$$\epsilon(u_{ix}) = \frac{1}{2}u_{ix}^2 + w(u_{ix}). \tag{5}$$

Here, the first term is the kinetic energy of the longitudinal motion and $w(u_{ix})$ is the enthalpy given by

$$w = \frac{\gamma}{(\gamma-1)}\frac{p_e}{m_i n_e} \propto u_{ix}^{-(\gamma-1)}, \tag{6}$$

which is related to E_x by

$$eE_x = -m_i\frac{dw}{dx},$$

in which we assume adiabatic flow with $p_e \propto n_e^\gamma \propto u_{ex}^{-\gamma} \propto u_{ix}^{-\gamma}$, where for charge neutrality, we have $n_{ex} = n_{ix}$ and $u_{ex} = u_{ix}$.

Now with u_{iz} given in terms of u_{ix} (through longitudinal momentum conservation) in Eq. (1) and with u_{iy} eliminated through the x component of the equation of motion

$$u_{ix}\frac{du_{ix}}{dx} = \frac{e}{m_i}(E_x + u_{iy}B_{0z}), \tag{7}$$

the conservation of energy integral, Eq. (4), becomes a first-order differential equation for the longitudinal flow speed u_{ix}. In normalized form, the equation may be written as

$$\frac{1}{2}\left[\left(1 - \frac{1}{M^2u^{\gamma+1}}\right)u\frac{du}{dx}\right]^2 = [P(u) - e(u)]\sin^2\theta - \frac{1}{2}P^2(u)\cos^2\theta \equiv E(u) \tag{8}$$

using the normalized variables $u = u_{ix}/U$, $x = x/l$ and $l = U/\Omega$. The normalized Bernoulli momentum, $P(u)$, and energy, $e(u)$, functions are given by

$$P(u) = u - 1 + \frac{1}{\gamma M^2}\left(\frac{1}{u^\gamma} - 1\right) \tag{9}$$

and

$$e(u) = \frac{1}{2}(u^2 - 1) + \frac{1}{(\gamma-1)M^2}\left(\frac{1}{u^{\gamma-1}} - 1\right), \tag{10}$$

where M is the Mach number of the wave, given by

$$M = \frac{U}{c}, \qquad c = \sqrt{\gamma kT_e/m_i}. \tag{11}$$

Equations (9) and (10) exhibit the adiabatic, thermodynamic relation $de = u\,dP$. In defining P and e, we have adjusted the constants in the conservation laws of Eqs. (1) and (4) to the undisturbed conditions at $x = -\infty$. Thus, $P(u)$ measures the change in the transition of the momentum, with the first and second terms on the right hand side, respectively, representing the change in dynamic pressure and the change in electron pressure. Similarly, in the definition of $e(u)$, the first term is the change in longitudinal kinetic energy and the second, the change in enthalpy. The "structure function" $E(u)$, defined by the right hand side of Eq. (8), combines both constants of the motion and its zeros define possible equilibrium points and the soliton amplitude.

III. SOLITON SOLUTION (SUBSONIC CASE, $M<1$)

The classical soliton solution of Eq. (8) has been discussed elsewhere (McKenzie[6]). The necessary condition for this solution follows from the requirement that $E(u)$, as defined by the right hand side of Eq. (8), possesses a double positive zero at the "initial" (equilibrium) point $u=1$, which occurs for M in the range

$$1 > M > \cos\theta. \tag{12}$$

This is equivalent to requiring that the linear stationary waves, with wave number k at an angle θ to B_0, where

$$k^2 = \frac{\Omega^2}{U^2}\frac{(M^2-\cos^2\theta)}{(M^2-1)}, \tag{13}$$

be evanescent, i.e., $k^2<0$, which is indeed the case when Eq. (12) is satisfied. In the weakly nonlinear regime, where $u=1+\delta$, $\delta \ll 1$, the Bernoulli momentum and energy functions, Eqs. (9) and (10), may be expanded around the initial point $u=1$ to yield the wave structure equation in the form

$$\frac{(M^2-1)}{M^2}\frac{d\delta}{dx} = \pm\frac{\delta}{M}\left[(1-M^2)(M^2-\cos^2\theta)-(\gamma+1)M^2\delta \times\left\{\cos^2\theta\left(\frac{2}{3}-\frac{1}{M^2}\right)+\frac{1}{3}\right\}\right]^{1/2}. \tag{14}$$

This weakly nonlinear form admits the classical sech[2] hump of compression of amplitude δ_m given by

$$\delta_m = [(1-M^2)(M^2-\cos^2\theta)]\Big/\left[(\gamma+1)M^2\left\{\cos^2\theta\left(\frac{2}{3}-\frac{1}{M^2}\right)+\frac{1}{3}\right\}\right], \tag{15}$$

in which the length scale is simply $|k^{-1}|$. [Equation (15) corrects a mistake for δ_m in the original paper (McKenzie *et al.*[7])]

In general, the amplitude of the solitary wave is given by the compressional root $u=u_c<1$ of the total energy function $E(u)$ as shown in Fig. 1(a). There is also a possible rarefactive equilibrium point $u=u_r>1$, which, however, cannot be reached since the sonic point $u=u_s=1/[M^{2/(\gamma+1)}]\,(<u_r)$ intervenes and prevents the formation of a smooth soliton because $du/dx\to\infty$ at this sonic point (which corresponds to choked flow). The wave amplitude of the soliton in the fully nonlinear case is given by the compressive solutions for u of the roots of the $E(u)=0$. These are given by

$$m^2 = \frac{1}{M^2} = \left[\frac{2p}{\gamma}+\frac{2ft^2}{\gamma(\gamma-1)} \pm\sqrt{\frac{2p}{\gamma}+\frac{2ft^2}{\gamma(\gamma-1)}-4(1+t^2)\frac{p^2}{\gamma^2}}\right]\Big/\left[\frac{2p^2}{\gamma^2}\right] \tag{16a}$$

$$= \frac{2u^2}{(u+1)^2}\left(u+1+t^2\pm t\sqrt{1+t^2-u^2}\right),\quad \gamma=2, \tag{16b}$$

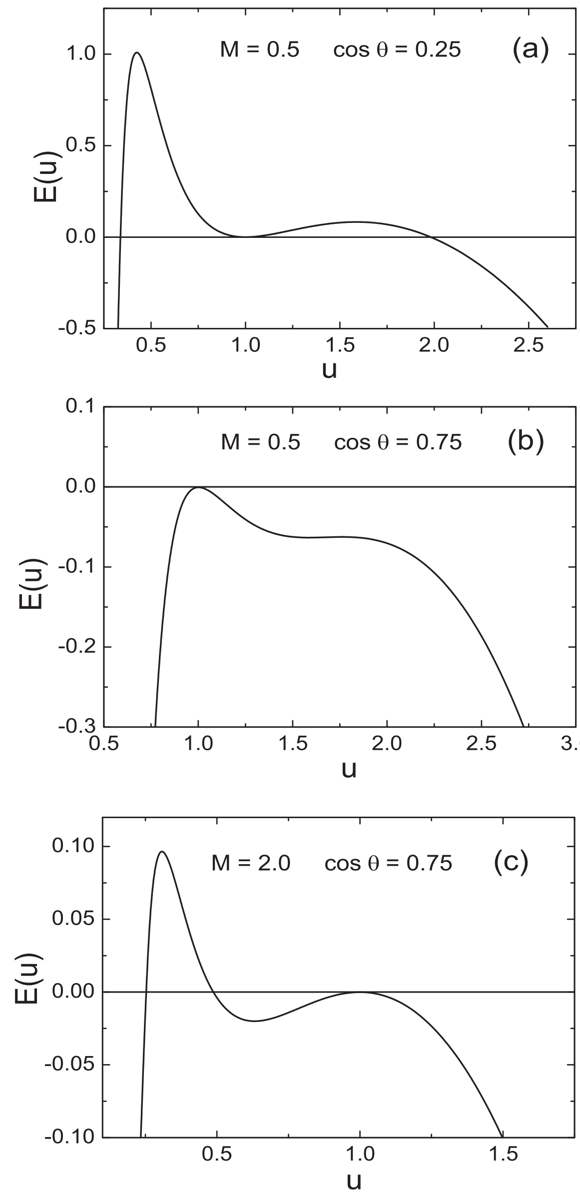

FIG. 1. Graph of $E(u)$ for three cases. (a) Subsonic ($M<1$) and sufficiently oblique waves yield compressive solutions. The wave amplitude is the compressional root $u=u_c(<1)$ of $E(u)=0$; (b) $M<1$ and quasi-parallel, $E(u)<0$ and periodic waves can be constructed with amplitudes given by the intersections of $E(u)$ with the line E_0, representing a driver field; (c) $M>1$ periodic waves can be constructed with amplitudes $u_c(<1)$ and $u_r(>1)$, provided $u_c>u_s$, the sonic point.

in which $t=\tan\theta$ and p, f, and g are given, respectively, by

$$p(u) = \frac{(u^\gamma-1)}{u^\gamma(u-1)} \quad \left(=\frac{u+1}{u^2}\text{ for }\gamma=2\right),$$
$$f(u) = \frac{-(\gamma-1)p+\gamma g}{(u-1)} \quad \left(=\frac{1}{u^2}\text{ for }\gamma=2\right),$$
$$g(u) = \frac{(u^{\gamma-1}-1}{u^{\gamma-1}(u-1)} \quad \left(=\frac{1}{u}\text{ for }\gamma=2\right). \tag{17}$$

The solutions of Eqs. (16b) are given in Fig. 2 as $m(u)$ for given $t=\tan\theta$, and Fig. 3 as $u(M)$ curves. The relevant solution for the amplitude of the soliton is, for a given $M<1$, the compressive solution $u<1$. This solution

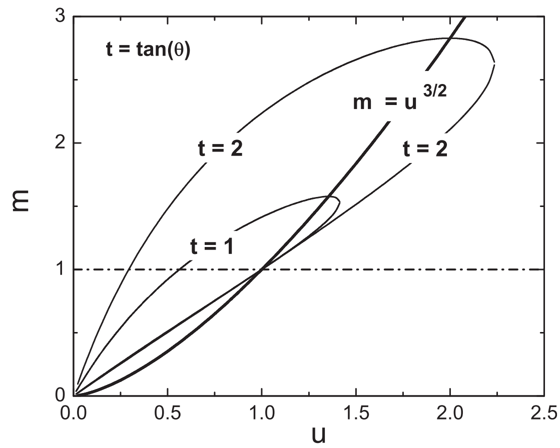

FIG. 2. Exact solutions for $m(u)$ from Eq. (16a). The (m, u) curves are given for $t = \tan\theta = 1$ and $t = 2$. Also shown is sonic curve $m = u^{3/2}$. Soliton solutions lie in the region $m > 1$ and $u < 1$.

demonstrates that the amplitude increases with decreasing M and increasing θ.

IV. PERIODIC SOLUTIONS

Nonlinear periodic waves are described by Eq. (8) when modified by the addition of a constant, E_0 to the right hand side of the equation. This constant may be regarded as a measure of the energy of an initial "driver" about the mean state

$$E(u) \to E(u) - E_0. \tag{18}$$

The amplitudes of possible periodic solutions of the wave structure equation are, therefore, given by the roots of $E(u) = E_0$ which are readily obtainable graphically from the intersections of the horizontal line E_0 with the curve of $E(u)$. The two possible cases, namely, subsonic ($M < 1$) and supersonic ($M > 1$) waves, for which their linear counterparts are indeed periodic (propagating) waves will now be treated.

A. Subsonic case ($M < \cos\theta$)

In this case, $E(u)$ has a double negative zero at the initial point (corresponding to propagating linear waves). The curve

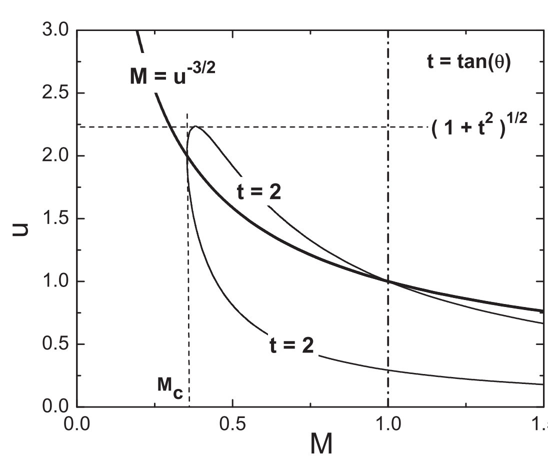

FIG. 3. The (u, M) solutions at given t. Soliton wave amplitudes lie in the domain $M < 1$ and $u < 1$.

of $E(u)$ has the form shown in Fig. 1(b) which manifests two extrema in the rarefactive regime $(u > 1)$. The maximum occurs at $u = u_+$, which in the case of $\gamma = 2$, is given by

$$u_+ = \frac{(\sqrt{1 + 8M^2/\cos^2\theta} - 1)}{4M^2/\cos^2\theta}. \tag{19}$$

There is also a maximum at $u = u_s$, where

$$u_s = \frac{1}{M^{2/(\gamma+1)}}, \tag{20}$$

so that $u_s > u_+$ and u_s lies to the right of the minimum. Hence, if $|E_0| < |E(u)|$, there are two intersections yielding a rarefactive root $u_r(<u_+)$ and a compressive root $u = u_c$, which yield the amplitudes of the wave. There exists a distinct asymmetry in the wave with the rarefactive amplitude exceeding the compressive amplitude. Moreover, an interesting limiting case arises when the "driver energy" attains the critical value

$$E_0 = E(u_+), \tag{21}$$

and E_0 touches the $E(u)$ curve at this point where $dE/du = 0$. A Taylor expansion of $E(u)$ around this point shows that the wave structure equation approximates to

$$\left[1 - \left(\frac{u_s}{u_+}\right)^{\gamma+1}\right] u_+ \frac{du}{dx} = \pm(u - u_+)\sqrt{E''(u_+)}. \tag{22}$$

Thus, $u \to u_+$ exponentially over the length scale defined by the above equation. In this limiting case, the "periodic" wave form looks like a soliton of compression (u_c) in which the "equilibrium" rarefactive amplitude is u_+.

B. Supersonic case ($M > 1$)

In the supersonic case ($M > 1$), $E(u)$ has a double negative zero at the initial point $u = 1$. The curve of $E(u)$ takes the form shown in Fig. 1(c) and in more detail in Fig. 4 for

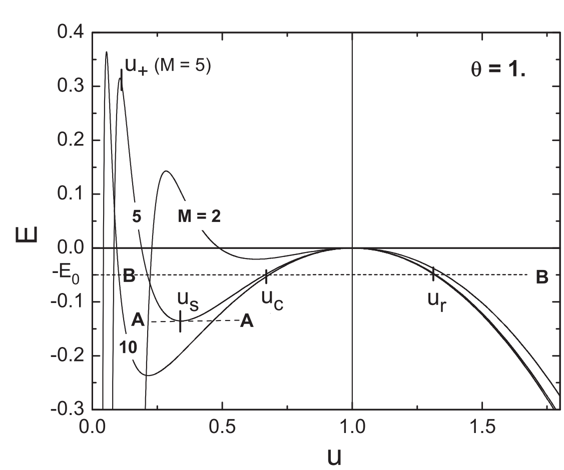

FIG. 4. The intersections between the line BB and $E(u)$ give the amplitudes of the periodic waves. The line AA represents the critical case where E_0 touches $E(u)$ at the sonic point, giving rise to a wedge shaped waveform at the compressive limiting amplitude.

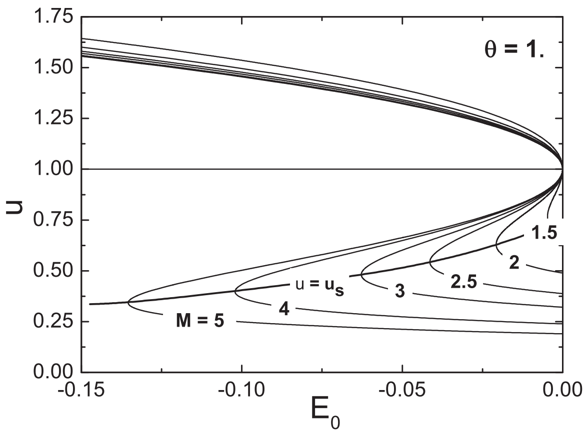

FIG. 5. The rarefactive and compressive wave amplitudes (solid curve) as a function of the driver field E_0 for various wave Mach numbers. The curve $u = u_s$ represents the limiting wave compressive amplitude for different values of M.

various M. In this case, there are two extrema; in the compressive phase, at $u = u_s$ and at $u = u_+$, where $u_+ < u_s$. Again the amplitudes of the periodic waves corresponding to a driver energy E_0 are given by the intersections of the horizontal line representing E_0 with the $E(u)$ curve, shown in Fig. 4 as $u_r(>1)$ and $u_c(<1)$. The extremum at $u = u_s$ corresponds to the local sonic point of the flow, at which steady flow becomes choked. This point is illustrated by the horizontal E_s line in Fig. 4 touching the $E(u)$ curve for $M = 5$ at the sonic point $u = u_s$. Solutions of the equation $E(u) = E_0$ are shown in Fig. 5 for different values of M at a given θ as (u, E_0) curves. Now when E_0 attains the value $E(u_s)$, the compressive root $u = u_c$ becomes sonic ($u_c = u_s$) and the numerator and denominator of the wave structure equation for du/dx both tend to zero $\propto (u - u_s)$ with the result that the slope at the critical transonic point becomes

$$\frac{du}{d(x/l)} = \pm\sqrt{E(u_s)}/(\gamma+1)$$
$$= \pm\sqrt{[(1-u_s^{-1})\sin^2\theta - \cos^2\theta P(u_s)u_s^{-1}]}/(\gamma+1). \quad (23)$$

The wave form therefore develops a wedge shape at this critical (sonic) point. Previously, this was described as a spiky or cusp-like waveform in which the slope at $u = u_s$ tended to infinity (McKenzie[6]), but this is, in fact, not the case as the zero of the transonic denominator is cancelled by the zero of the numerator. This is immediately apparent from a Taylor expansion of $E(u) - E_0$ around the minimum of $E(u)$ at $u = u_s$. Similar wedge shaped waveforms appear in the analysis of water waves, including a Coriolis term (Shrira[8] and McKenzie[9]). This latter behaviour obtains for precisely the same reason as has been given here, namely, that a wedge shape forms when the flow becomes critical. In the case of water waves, this implies that the local flow speed matches the local shallow water phase speed, whereas, in the case of the electrostatic ion-cyclotron wave, the critical flow speed is the ion-acoustic speed.

The analytic expressions for these solutions are given by Eq. (16b) except that in the radical we multiply $4c^2p^2/\gamma^2$ by $[1 + 2E_0/(u-1)^2]$, which follows directly from solving $E(u) = E_0$. Figure 5 shows the amplitudes of the periodic waves as the rarefactive root and the compressive root, the latter is less (or equal) to u_s, both of which increase with Mach number M.

V. SUMMARY

Two fundamental constants of motion of the wave system (parallel momentum and total energy) reduce the wave structure equation to a first order ordinary differential equation for the longitudinal flow speed in the transition. Compressive soliton solutions exist for waves propagating at subsonic speeds oblique to the magnetic field. In the present treatment an exact expression for the wave amplitude as a function of the wave Mach number M and its obliquity θ is given as shown in Fig. 3. In the case of supersonic wave speeds, periodic wave structures can be constructed, whose compressive and rarefactive amplitudes are also given by a similar exact expression, which now includes the effect of the initial driver field. The asymmetry about the compressive and rarefactive phases of the wave is illustrated in Fig. 4, which also demonstrates the formation of a wedge shaped wave form brought about by the gas dynamics associated with transonic flow.

ACKNOWLEDGMENTS

J.F.M. wishes to thank the Durban University of Technology and J.F.M. and T.B.D wish to thank the National Research Foundation of South Africa (under Grant No. 2053776) for support.

[1]R. V. Reddy, G. S. Lakhina, N. Singh, and R. Bharuthram, Nonlinear Processes Geophys. **9**, 25–29 (2002).
[2]K. Shin, H. Kojima, H. Matsumoto, and T. Mukai, J. Geophys. Res., [Space Phys.] **113**, AO3101, doi:10.1029/2007JA012344 (2008).
[3]J. D. Richardson, J. C. Kasper, C. Wang, J. W. Belcher, and A. J. Lazarus, Nature **454**, 63–66 (2008).
[4]R. H. Burrows, G. P. Zank, G. M. Webb, L. F. Burlaga, and N. F. Ness, Astrophys. J. **715**, 1109–1116 (2010).
[5]L. B. Wilson III, C. Cattell, P. J. Kellog, K. Kersten, L. Hanson, R. MacGregor, and J. C. Kasper, Phys. Rev. Lett. **99**, 041101 (2007).
[6]J. F. McKenzie, J. Plasma Phys. **70**, 533–541 (2004).
[7]J. F. McKenzie, E. Dubinin, K. Sauer, and T. B. Doyle, J. Plasma Phys. **70**, 431–462 (2004).
[8]V. I. Shrira, Izv. Atmos. Oceanic Phys. **22**(4), 395–405 (1986).
[9]J. F. McKenzie, Quaest. Math. **33**(4), 409–416 (2010).
[10]M. Temerin, M. Wolderif, and P. S. Mozer, Phys. Rev. Lett. **43**, 1941–1943 (1979).
[11]M. Y. Yu, P. K. Shukla, and S. Bujarburua, Phys. Fluids **23**, 2146–2147 (1980).
[12]P. Jovanic and P. K. Shukla, Phys. Rev. Lett. **84**, 4373–4376 (2000).
[13]J. F. McKenzie, Phys. Plasmas **9**, 800–805 (2002).
[14]J. F. McKenzie and T. B. Doyle, Phys. Plasmas **9**, 55–65 (2002).